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MARTINGALES IN MARK-RECAPTURE EXPERIMENTS  
WITH CONSTANT RECRUITMENT AND SURVIVAL

A  
THESIS

Presented to the Faculty  
of the University of Alaska Fairbanks  
in Partial Fulfillment of the Requirements  
for the Degree of

DOCTOR OF PHILOSOPHY

By  
Patricia B. Humphrey, B.A., M.A.

Fairbanks, Alaska

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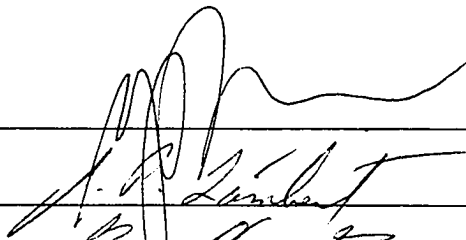
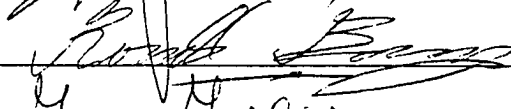
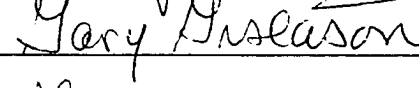

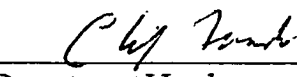
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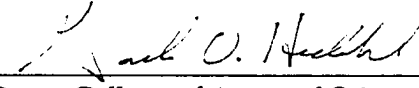
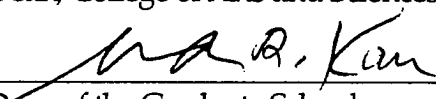
By

Patricia B. Humphrey

RECOMMENDED:

  
A. J. Lambert  
  
Robert Brown  
  
Gary Hulseason  
  
Pham X. Quang  
Advisory Committee Chair  
  
Chy Tando  
Department Head

APPROVED:

  
Paul O. Hedderley  
Dean, College of Arts and Sciences  
  
M. R. Kan  
Dean of the Graduate School  
6-21-95  
Date

## MARTINGALES IN MARK-RECAPTURE EXPERIMENTS WITH CONSTANT RECRUITMENT AND SURVIVAL

### ABSTRACT

The method known as mark-recapture has been used for almost one hundred years in assessing animal populations. For many years, these models were restricted to closed populations; no changes to the population were assumed to occur through either migration or births and deaths. Numerous estimators for the closed population have been proposed through the years, some of the most recent by Paul Yip which make use of martingales to derive the necessary estimates.

The independently derived Jolly-Seber model (1965) was the first to address the open population situation. That method as originally proposed is cumbersome mathematically due to the large number of parameters to be estimated as well as the inability to obtain estimates until the end of a series of capture events since some of the "observed" variables necessary are prospective. It also is cumbersome for the biologist in the field as individual marks and capture histories are required for each animal. Variations have been proposed through the years which hold survival and / or capture probabilities constant across capture occasions. Models based on log-linear estimators have also been proposed (Cormack 1989).

This paper builds on the closed population work of Yip in using martingale-based

conditional least squares to estimate population parameters for an open population where it is assumed recruitment of new individuals into the population is constant from one capture occasion to the next, and capture and survival probabilities are constant across capture occasions. It is an improvement over most other methods in that no detailed capture histories are needed; animals are simply noted as marked or unmarked.

Performance of the estimator proposed is studied through computer simulation and comparison with classical estimators on actual data sets.

## TABLE OF CONTENTS

List of Figures .....	6
List of Tables .....	7
Chapter 1 .....	8
1.1 Estimation of a Closed Population .....	8
1.2 Estimation of an Open Population .....	13
1.3 Comparison of Proposed Model with Others .....	15
Chapter 2 .....	17
2.1 Assumptions of our Model .....	17
2.2 Notation .....	18
2.3 Martingales .....	19
2.4 Conditional Least Squares .....	20
2.5 Basic Properties of our Model .....	20
Chapter 3 .....	25
3.1 Probability Generating Functions for $M_i$ , $U_i$ , $m_i$ and $u_i$ .....	25
3.2 Asymptotic Consistency .....	29
3.3 Asymptotic Normality of Estimates .....	45
Chapter 4 .....	50
4.1 Simulation Results.. ..	51
4.2 Comparison with other estimators .....	58
Chapter 5 .....	63
References .....	65
Appendix A. Moments of $M_i$ and $U_i$ .....	68
Appendix B. FORTRAN Estimation and Bootstrap Code .....	74



## LIST OF FIGURES

Figure 4.1 Estimation of the initial population, $N$ . . . . .	53
Figure 4.2 Estimation of $\lambda$ . . . . .	54
Figure 4.3 Estimatin of $\phi$ . . . . .	55
Figure 4.4 Estimation of $p$ . . . . .	56
Figure 4.5 Effect of the number of capture occasions on estimation of $N$ and $\lambda$ . . . . .	57
Figure 4.6 Effect of the number of capture occasions on estimation of $p$ and $\phi$ . . . . .	58

## LIST OF TABLES

Table 4.1 Summary Statistics for Meadow Vole data .....	59
Table 4.2 Estimates for Meadow Vole data .....	60
Table 4.3 Summary Statistics for Gray Squirrel data .....	61
Table 4.4 Estimates for Gray Squirrel data .....	62

## CHAPTER 1

### 1.1 Estimation of a Closed Population.

The estimation of population sizes has been a topic of interest for many years. A technique now known as *mark-recapture*, or *capture-recapture*, has been used since Carl Petersen first published the pioneering work in 1896. He had invented a brass tag that was attached to plaice in order to study their migrations. When one-third of the tags were later retrieved by fishermen, he realized this information could be used to estimate the total population size (White, 1982). His estimator, now referred to as the Lincoln-Petersen, is based on a single marking event, with a single recapture. We make assumptions that marked and unmarked animals are equally likely to be caught, and no deaths, births (recruitment) or migration into or out of the population have occurred between the two capture occasions. In other words we have a *closed* population. Equating the two estimates of capture probabilities, this can be stated mathematically as

$$\frac{n_1}{\hat{N}} = \frac{m_2}{n_2}$$

where  $\hat{N}$  is the estimated population size;  $n_1$ ,  $n_2$  are the numbers caught on the first and second capture occasions; and  $m_2$  is the number of marked individuals caught on the second sampling occasion. This gives rise to the estimator

$$\hat{N} = \frac{n_1 n_2}{m_2} .$$

This estimator further is a maximum likelihood estimate if one conditions on  $N$ ,  $n_1$  and  $n_2$ . In that case, we have

$$P(m_2 | N, n_1, n_2) = \frac{\binom{n_2}{m_2} \binom{N-n_2}{n_2-m_2}}{\binom{N}{n_2}} .$$

Maximizing with respect to  $N$ , we derive the estimator. Chapman (1951) showed that while this estimator is the best asymptotically normal estimate it is, however, biased especially for small samples. He proposed the unbiased estimator

$$\hat{N} = \frac{(n_1 + 1)(n_2 + 1)}{m_2 + 1} - 1 .$$

Zoe Schnabel (1938) extended this estimator to situations where there were more than two capture occasions. Her estimator, again based upon conditioning on the total capture numbers, also involves assumptions that  $m_i \sim \text{bin}(n_i, p_i)$  where  $p_i = M_i/N$  and the ratio  $n_i/N$  is sufficiently small to overcome the problems of sampling without replacement. This leads to the likelihood equation

$$\mathcal{L}(m_1, m_2, \dots, m_t) = \prod_{i=1}^t \binom{n_i}{m_i} \left( \frac{M_i}{N} \right)^{m_i} \left( 1 - \frac{M_i}{N} \right)^{n_i - m_i}$$

with the associated maximum likelihood estimate being the appropriate root of

$$\sum_{i=1}^t \frac{(n_i - m_i) M_i}{N - M_i} = 0$$

which is

$$\hat{N} = \frac{\sum_{i=1}^t n_i M_i}{\sum_{i=1}^t m_i}.$$

The Schnabel model has become known in the literature as  $M_t$  since it allows capture probabilities to vary from occasion to occasion.

Many variations of the above have been proposed and treated for the closed population scenario, variously relaxing the assumption of equal capture probabilities. The model normally referred to as  $M_b$  allows for a behavioral response to capture (trap avoidance or fascination) by the animals. Finally, allowing each animal to have its own individual capture probability yields the model known as  $M_{th}$ . There are also various combinations of these models. Drawbacks to these models include the necessity for individual marks on each animal, the need for detailed capture history for each animal and

a lack of maximum likelihood estimates for many of the models. Some models as yet do not have estimators (Pollock et al 1990). One recent development in this arena is the derivation of an estimator for the  $M_{th}$  model, in which capture probabilities vary with time and animals. Chao et al (1992) derived a nonparametric estimator based on the idea of sample coverage, a proportion of the total individual capture probabilities of captured animals which can be viewed as a measure of the completeness of the sample.

Others have taken a Bayesian approach to the closed population problem (Zacks et al 1990 and Leite and Pereira 1990). Zacks obtained a Bayesian estimator based on a Poisson prior distribution. His final estimates of the total population are extremely close to the total number of unique individuals captured during the experiment. Leite and Pereira derive estimates based on an open-minded (non-informative) prior and show that the Bayes' estimates form a martingale to obtain asymptotic consistency of the estimates.

Sen (1987) reviewed several classical methods in the context of sequential estimates, and again uses martingale theory for asymptotic results. He also includes an estimator that involves a cost of animal capture, the point being to minimize a risk function by selecting an optimal number of individuals to be sampled.

A recent development in the area of closed population models has involved the use of martingales for direct estimation of the population size, rather than a conditional likelihood approach with martingales appealed to for asymptotic results. Becker (1984) and Yip (1989) used identical martingales in a discrete time setting to address the estimation problem. Both papers' estimators become identical to that of Schnabel.

Becker's 1984 paper includes a continuous time analogue in which capture probabilities are allowed to vary among individuals according to a Beta distribution. Yip later (1991) used the same martingale in a discrete time setting and derived a standard error for the estimate. Becker and Heyde (1990) revisit the continuous time martingale estimator, and examine asymptotic relative efficiency against a maximum likelihood estimator for three choices of weight functions.

Yip (1991) revisited the martingale of the 1989 paper with the addition of an "optimal" weight function in the manner of Godambe (1985). Yip has also proposed a modified martingale that addresses the case of known removals from the population (1991). Lloyd and Yip (1991) present a unification of the preceding martingale work and contrast the martingale approach with conditional likelihood. A version conditioned on the observed number of captures, in which the numbers of marked and unmarked are considered to have a hypergeometric distribution, has also been considered (1993). Finally, Yip, Fong and Wilson (1993) revisit the continuous time martingale setting, present an alternative estimator with associated standard error that may be of use in a situation with incomplete recording of capture histories, and present simulation results which contrast the martingale estimators with those of Darroch and Ratliff as well as maximum likelihood.

## 1.2 Estimation of an Open Population.

One drawback of a closed population model is clearly that if the capture occasions are not “close enough” together in time, mortality can occur, as well as new recruitment into the population by either immigration or birth. Even in tightly spaced experiments, a closed model may not be appropriate if the species of interest is sufficiently short-lived. Indeed, the dynamics of population change may be of more interest than the actual population numbers. The pioneering works in this area were developed and published independently by George Jolly and George Seber in 1965. The Jolly-Seber model, as it is now known, allows capture probabilities to vary with time, and estimates the capture probabilities as well as survival rate  $S_j$  from occasion  $j$  to  $j+1$ , total influx to the population  $B_j$ , and population levels  $N_j$  at occasions 2, 3, ...,  $(t-1)$ . The population size at time  $(j+1)$  is related to these parameters through the relationship  $N_{j+1} = N_j S_j + B_j$ .

Certain problems arise with estimates derived from the original Jolly-Seber model. Due to the large number of parameters, estimates are usually imprecise. It is impossible to estimate the final population as well as the survival rate for the final two periods. Certain of the “observed” variables are prospective in nature: the number seen prior to occasion  $j$  which are not seen at occasion  $j$  but seen later and the number of animals caught at occasion  $j$  which are later recaptured. In addition, it is possible to obtain improbable results: negative estimates for the recruitment parameters  $B_j$  often occur as well as survival probabilities greater than 1. In an attempt to address some of these concerns Jolly (1982) proposed variations, again based on conditional maximum likelihood, for situations



where one might assume capture probabilities, survival probabilities or both are constant across inter-capture intervals. He also proposed a  $\chi^2$  goodness of fit test for use in model selection. One should note that here again, negative estimates of the  $B_j$  parameters commonly occur. Brownie et al (1986) extended Jolly's goodness of fit test to make fuller use of the data by utilizing the individual capture histories. They also develop estimators and goodness of fit tests for a population studied as two distinct age classes.

Crosbie and Manly (1985) proposed a factorial series of models, again based on maximum likelihood, which allow for varying assumptions on capture probability, survival distributions and birth distributions. In this they consider the numbers of animals having a particular capture history to be a single multinomial distribution. Examining likelihood ratios among the various possibilities will lead to the most appropriate model with the fewest number of parameters.

Yet another approach toward parsimony was taken by McKay and Bradley (1988). Like Crosbie and Manly, they also consider the number of animals with a particular capture history as multinomially distributed, however, they allow for an Erlangian ( $\beta(1, n)$ ) distribution on both birth time and survival time. Their estimator is then based on a conditional (with respect to the total number of captured animals) maximum likelihood.

Cormack (1989) used the multinomial capture histories in yet a different way, using the theory of log-linear models to obtain estimates of expected counts for each observable capture history. These in turn are transformed into estimates of  $N$  and capture probabilities. He demonstrated the method for a closed population, a situation where trap

dependence occurs ( $M_{\pm}$ ), and for open populations. In the open population case, however, not all parameters of interest are estimable, a situation in keeping with the Jolly-Seber model, with which he claims exact equivalence.

### 1.3 Comparison of Proposed Model with Others.

This paper will propose a model based on martingales and the theory of conditional least squares for a special case of the open population scenario, namely allowing for constant survival rates between capture occasions, constant probability of capture across occasions and individuals, and a constant rate of recruitment between capture occasions. It is significantly different from the preceding in that it does not use a conditional likelihood approach. Individual capture histories (and thus individual marks) are not necessary, thus normally resulting in a smaller required field effort. The martingale we propose is unusual in that it possesses the martingale property with respect to a sigma field generated by unobservable variables.

In chapter 2 we will define notation, assumptions and present the proposed model. Background information on martingales and the method of conditional least squares will be included as well.

Chapter 3 contains results related to asymptotic properties of the estimates derived. We will show the model possesses the characteristics necessary to ensure strong consistency of the estimates, that is, they asymptotically converge to the true underlying values. In addition, we will show necessary and sufficient conditions for joint asymptotic

normality of the estimates are fulfilled.

Chapter 4 contains results and a discussion of simulation experiments that assess actual performance of the model.

Chapter 5 will present a general discussion of the model, and potential further work related to relaxing of assumptions.

## CHAPTER 2

In this chapter we outline the basic assumptions involved in the proposed model, introduce notation to be used throughout the remainder, and discuss basic information on martingales and the method of conditional least squares. We then present our proposed model.

### 2.1 Assumptions of our model.

Formally stated, our assumptions are these.

1. Every individual in the population has the same probability,  $p$ , of being captured on each occasion.
2. All capture events are mutually independent.
3. Each individual is considered to survive independently of others in the population. Every individual has the same probability of survival,  $\phi$ , from occasion  $i$  to occasion  $i + 1$ .
4. Recruitment is considered to take place immediately prior to a capture occasion.
5. There are no losses on capture. All caught animals are returned to the population.
6. No marks are lost and all marks are correctly identified.

## 2.2 Notation.

The following notation will be used throughout this paper.

$N$	the initial population level.
$\phi$	the probability of an individual surviving from time $i$ to $i+1$
$\psi$	$1 - \phi$ , the probability of not surviving from one occasion to the next
$p$	the probability of an individual to be caught on any capture occasion
$q$	$1-p$ , the probability of not being caught
$\lambda$	the recruitment rate of new individuals to the population at each capture after the first
$t$	the total number of capture occasions
$M_i$	the total number of marked individuals in the population immediately prior to occasion $i$ . This is not an observable quantity.
$U_i$	the total number of unmarked individuals in the population immediately prior to occasion $i$ which have survived from prior occasions. This, like $M_i$ , is unobservable.
$m_i$	the number of marked individuals captured at occasion $i$
$u_i$	the number of unmarked individuals captured at occasion $i$

### 2.3 Martingales.

Consider a sequence of random variables  $\{x_n\}$ , and an increasing sequence of sigma fields  $\mathcal{F}_n = \sigma(x_1, x_2, \dots, x_n)$  such that  $x_n$  is measurable with respect to  $\mathcal{F}_n$ .

Define  $X_n = \sum_{j=1}^n x_j$  as a partial sum of the random variables. We denote the expected value of  $X_n$  conditional on the history of the process, as  $\mathcal{E}(X_n | \mathcal{F}_{n-1})$ .

Definition: A sequence of random variables and Borel fields  $\{X_n, \mathcal{F}_n\}$  is called a martingale if and only if we have for each  $n$ :

1.  $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ ;
2.  $X_n$  is measurable with respect to  $\mathcal{F}_n$ ;
3.  $\mathcal{E}(X_{n+1} | \mathcal{F}_n) = X_n$ .

We have a *zero-mean martingale* if, in addition to the above,  $\mathcal{E}(X_1) = 0$ . The martingale differences  $h_1, h_2, \dots, h_n$  are defined by  $h_0 = 0, h_n = X_n - X_{n-1}$ . From property 3 above, we have the implication that  $X_n$  is a martingale if  $\mathcal{E}(h_i | \mathcal{F}_{i-1}) = 0$  for all  $i$ .

One example of a martingale is the cumulative fortune,  $X_n$ , of a gambler in a fair game of coin tossing. If heads and tails are equally likely, with equal gain or loss,  $x_i$ , on each toss of the coin, we have

$$\mathcal{E}(X_{n+1} | \mathcal{F}_n) = \mathcal{E}(X_n + x_{n+1} | \mathcal{F}_n) = X_n + \mathcal{E}(x_{n+1} | \mathcal{F}_n) = X_n$$

and the martingale property is verified.

## 2.4 Conditional Least Squares.

In ordinary least squares, the sum of squared deviations of independent observations from their expected values is minimized, subject to certain assumptions. In stochastic processes observations are not independent, so applying this method would be incorrect. Klimko and Nelson (1978) introduced the method of conditional least squares which can be used in a situation where observations are dependent. Hall and Heyde (1980) quote extensively from Klimko and Nelson in their discussion of conditional least squares as it applies to martingales.

Conditional least squares differs from the ordinary case in that we minimize the sum of squared deviations of dependent observations from their conditional expectations. While the method is not optimal in the sense of “best linear unbiased estimates,” it yields estimates of parameters, which, under certain conditions are strongly consistent and jointly asymptotically normally distributed. That is, estimates are asymptotically unbiased and normal quantiles may be used in forming confidence intervals.

## 2.5 Basic properties of our model.

All individuals are considered to be independent of each other. Conditional on the prior history, we have the following.

1.  $m_i$  and  $u_i$  are independent of each other, since we can consider these to be sampled from two distinct populations.
2.  $m_i$  has a binomial distribution with parameters  $M_i$  and  $p$ ; that is,  $m_i \sim \text{Bin}(M_i, p)$ .

Similarly,  $u_i \sim \text{Bin}((U_i + \lambda N), p)$  for  $i \geq 2$ ;  $u_1 \sim \text{Bin}(N, p)$ .

3.  $M_i \sim \text{Bin}((M_{i-1} + u_{i-1}), \phi)$ .  $U_i \sim \text{Bin}((U_{i-1} + \lambda N - u_{i-1}), \phi)$  for  $i \geq 2$ ;  
 $U_2 \sim \text{Bin}(N - u_1, \phi)$ . Note that  $M_i$  and  $U_i$  are unobservable.

We will define a martingale with respect to the following sequence of nested sigma fields:  $G_1 =$  the trivial sigma field,  $\mathcal{F}_1 = \sigma(u_1)$ ,  $G_2 = \sigma(u_1, U_2, M_2)$ ,  $\mathcal{F}_2 = \sigma(u_1, U_2, M_2, u_2, m_2)$ , ...,  $G_i = \sigma(u_1, \dots, U_i, M_i)$ ,  $\mathcal{F}_i = \sigma(u_1, \dots, U_i, M_i, u_i, m_i)$ . The  $\sigma$ -fields  $G_i$  represent the history of the process until just prior to the  $i$ -th capture occasion, while the  $\mathcal{F}_i$  represent the history up to and including the  $i$ -th capture event. We will denote conditional expectations with respect to these  $\sigma$ -fields as  $\mathcal{E}(x | G_i) = \mathcal{E}_i^-(x)$  and  $\mathcal{E}(x | \mathcal{F}_i) = \mathcal{E}_i(x)$ .

We first observe the following, making use of the Tower property of conditional Expectation (Chung 1974 p. 304):

$$\begin{aligned} \mathcal{E}_{i-1}(u_i) &= \mathcal{E}_{i-1}(\mathcal{E}_i^-(u_i)) \\ &= \mathcal{E}_{i-1}(p(U_i + \lambda N)) \\ &= p\lambda N + p\mathcal{E}_{i-1}(U_i) \\ &= p\lambda N + p\phi(U_{i-1} + \lambda N + u_{i-1}). \end{aligned}$$

We also have

$$\begin{aligned} \mathcal{E}_{i-1}(m_i) &= \mathcal{E}_{i-1}(\mathcal{E}_i^-(m_i)) \\ &= \mathcal{E}_{i-1}(pM_i) \\ &= p\phi(M_{i-1} + u_{i-1}). \end{aligned}$$



We wish to claim that the quantities

$$\begin{aligned} h_i &= m_i - u_i - \phi m_{i-1} + \phi(1-2p)u_{i-1} + p\lambda N ; \quad i \geq 2 \\ h_1 &= u_1 - Np \end{aligned}$$

are zero-mean martingale differences with respect to the sigma fields  $G_i$  and therefore the sum of these quantities is a zero-mean martingale.

Lemma 2.1. With respect to the sigma-fields  $G_{i-1}$ ,  $\sum_{i=1}^t h_i$  is a zero-mean martingale.

Proof.

Clearly  $\mathcal{E}(h_1) = 0$ . In order to verify the martingale property, we really need to show  $\mathcal{E}_{i-1}(h_i) = 0$ . Using the Tower property again, we have  $\mathcal{E}_{i-1}(h_i) = \mathcal{E}_{i-1}(\mathcal{E}_{i-1}(h_i))$ .

Also, with respect to the  $\sigma$ -field  $\mathcal{F}_{i-1}$ , both  $u_{i-1}$  and  $m_{i-1}$  are known (and therefore constant). Also, we have

$$\begin{aligned} \mathcal{E}_{i-1}(m_i) &= \mathcal{E}_{i-1}(p\phi(M_{i-1} + u_{i-1})) \\ &= p\phi M_{i-1} + p^2\phi(U_{i-1} + \lambda N) \end{aligned}$$

$$\begin{aligned} \mathcal{E}_{i-1}(u_i) &= p\lambda N + p\phi\mathcal{E}_{i-1}(U_{i-1} + \lambda N - u_{i-1}) \\ &= p\lambda N + p\phi U_{i-1} + p\phi\lambda N - p^2\phi(U_{i-1} + \lambda N) \\ &= p\lambda N(1 + \phi) + p\phi(U_{i-1} - p(U_{i-1} + \lambda N)). \end{aligned}$$

We make use of these to find

$$\begin{aligned}
 \mathcal{E}_{i-1}'(\mathcal{E}_{i-1}(h_i)) &= \mathcal{E}_{i-1}'(p\phi(M_{i-1} + u_{i-1}) - p\phi(U_{i-1} + \lambda N - u_{i-1}) - p\lambda N - \phi m_{i-1} \\
 &\quad + (1 - 2p)\phi u_{i-1} + p\lambda N) \\
 &= \mathcal{E}_{i-1}'(p\phi M_{i-1} - p\phi U_{i-1} - p\phi \lambda N - \phi m_{i-1} + \phi u_{i-1}) \\
 &= p\phi M_{i-1} - p\phi U_{i-1} - p\phi \lambda N - p\phi M_{i-1} + p\phi(U_{i-1} + \lambda N) \\
 &= 0.
 \end{aligned}$$

This proves lemma 2.1.

Using the martingale differences, we can form a conditional least squares objective function

$$\begin{aligned}
 Q_t(\theta) &= \sum_{i=1}^t [m_i - u_i - \mathcal{E}_{i-1}'(m_i - u_i)]^2 = \sum_{i=1}^t h_i^2 \\
 &= \sum_{i=1}^t [m_i - u_i - \phi m_{i-1} + \phi(1 - 2p)u_{i-1} + \lambda N p]^2
 \end{aligned}$$

in the manner of Klimko and Nelson (1978) and Hall and Heyde (1980). Here we have

$\theta = (p, \phi, N, \lambda)'$ . We will find  $\hat{\theta}$  by minimizing  $Q_t(\theta)$  with respect to  $\theta$ . Since we have a series of dependent random variables,  $\mathcal{E}(m_i - u_i | G_{i-1})$  is the best predictor of  $m_i - u_i$  given the past. Conditional least squares will seek to minimize the mean prediction error.

Estimates for  $\theta$  will be roots of the system of equations

$$\frac{\partial Q_t(\theta)}{\partial \theta_i} = 0 ; \quad i=1,2,3,4.$$

$$\frac{\partial Q_t(\theta)}{\partial p} = N(-u_1 + Np) + \sum_{i=2}^t (m_i - u_i - \phi m_{i-1} + (1 - 2p)\phi u_{i-1} + \lambda Np)(\lambda N - 2\phi u_{i-1})$$

$$\frac{\partial Q_t(\theta)}{\partial \lambda} = Np \sum_{i=2}^t (m_i - u_i - \phi m_{i-1} + (1 - 2p)\phi u_{i-1} + \lambda Np)$$

$$\frac{\partial Q_t(\theta)}{\partial N} = p(-u_1 + Np) + \lambda p \sum_{i=2}^t (m_i - u_i - \phi m_{i-1} + (1 - 2p)\phi u_{i-1} + \lambda Np)$$

$$\frac{\partial Q_t(\theta)}{\partial \phi} = \sum_{i=2}^t (m_i - u_i - \phi m_{i-1} + (1 - 2p)\phi u_{i-1} + \lambda Np)(2pu_{i-1} - m_{i-1})$$

In practice, estimates for  $\theta$  will normally be found using a numerical algorithm.

Explicit estimates for the variance of parameter estimates are not presented here.

In practice bootstrap estimation would be appropriate. The bootstrap method involves first estimating the parameters of interest then performing a computer simulation which duplicates the original experiment based on the assumed distributions of the observed

variables and the parameter estimates. Variances of the originally estimated parameters are then approximated by the observed variance from the simulation.

## CHAPTER 3

This chapter will explore the mathematical properties of the model proposed. We first demonstrate an unconditional joint probability generating function for the numbers of marked and unmarked animals both in the population and in a capture sample. Asymptotic consistency of the estimates obtained will then be addressed. Asymptotic normality of the estimates will be shown through the Cramèr-Wold technique through a version of the martingale central limit theorem.

### 3.1 Probability Generating Functions for $M_i$ , $U_i$ , $m_i$ and $u_i$ .

Define function  $K(x) = 1 - \phi + \phi x$ .  $K(x)$  is clearly the probability generating function for the binomial survival process across a single time period. We have  $K_1(x) = 1 - \phi + \phi x$ . By composition,  $K_2(x) = K(K(x)) = 1 - \phi + \phi(1 - \phi + \phi x) = 1 - \phi^2 + \phi^2 x$ . Repeating the composition of functions, we have  $K_j(x) = 1 - \phi^j + \phi^j x$ . This represents the fact that an animal survives  $j$  periods with probability  $\phi^j$  and does not survive with probability  $1 - \phi^j$ .

Similarly, define  $G(x, y) = px + qy$ . Here,  $p = 1 - q$  and  $q$  are binomial probabilities. We have  $G_1(x, y) = (1 - q)x + qy$ . This function can be thought of as relating to the capture process and transition of an animal from unmarked to marked. An animal only remains in the unmarked ( $y$ ) state if it is uncaught. Once caught, with probability  $p$ , it will change to a marked ( $x$ ) state. Define composition of the functions  $G$  as follows.

$G_2(x, y) = G(G(x, y)) = G(x, G(x, y)) = (1 - q)x + q((1 - q)x + qy) = (1 - q^2)x + q^2y$ . Again,

repeating the process, we have  $G_j(x, y) = (1 - q^j)x + q^jy$ .

Define composition of the functions  $K$  and  $G$  as follows.

$$G(K_j) = G(K_j(x), K_j(y)) = p(1 - \phi^j + \phi^jx) + q(1 - \phi^j + \phi^jy) = 1 - \phi^j + \phi^j(px + qy) = K_j(G(x, y)).$$

$$\text{Thus, } G(K_j(G(x, y))) = pK_j(x) + qK_j(G(x, y)) = p(1 - \phi^j + \phi^jx) + q(1 - \phi^j + \phi^j(px + qy)).$$

$$\text{Further simplification yields } G(K_j(G(x, y))) = 1 - \phi^j + \phi^j((1 - q^2)x + q^2y) = K_j(G_2(x, y)).$$

$$\text{Again, repeat the process to obtain } K_j(G_j(x, y)) = 1 - \phi^j + \phi^j[(1 - q^j)x + q^jy].$$

Now, define function  $f(x, y) = \mathcal{E}(x^{M_i} y^{U_i})$ . Making use of conditional expectations, we have  $f(x, y) = \mathcal{E}(\mathcal{E}_{i-1}(x^{M_i} y^{U_i})) = \mathcal{E}(\mathcal{E}_{i-1}(x^{M_i}) \mathcal{E}_{i-1}(y^{U_i}))$  since  $M_i$  and  $U_i$  are independent conditional on  $\mathcal{F}_{i-1}$ . Also, conditional on  $\mathcal{F}_{i-1}$ , we have  $M_i$  distributed as

$\text{Bin}(M_{i-1} + u_{i-1}, \phi)$  and  $U_i$  distributed as  $\text{Bin}(U_{i-1} + \lambda N - u_{i-1}, \phi)$ . Substituting

$K(x) = 1 - \phi + \phi x$  for  $x$ , and  $K(y) = 1 - \phi + \phi y$  for  $y$ , we have

$$f(x, y) = \mathcal{E}[(1 - \phi + \phi x)^{M_{i-1} + u_{i-1}} (1 - \phi + \phi y)^{U_{i-1} + \lambda N - u_{i-1}}] \quad (1)$$

$$f(x, y) = [K(y)]^{\lambda N} \mathcal{E} \left[ (K(x))^{M_{i-1}} (K(y))^{U_{i-1}} \left[ \frac{K(x)}{K(y)} \right]^{u_{i-1}} \right] \quad (2)$$

But, in equation (2), we have, again using conditional expectations,

$$\mathcal{E}_{(i-1)} \left[ \frac{K(x)}{K(y)} \right] = \left[ q + p \frac{K(x)}{K(y)} \right]^{U_{i-1} + \lambda N} = \left[ \frac{qK(y) + pK(x)}{K(y)} \right]^{U_{i-1} + \lambda N} = \left[ \frac{K(G(x, y))}{K(y)} \right]^{U_{i-1} + \lambda N}$$

Substituting into equation (2), we have

$$\begin{aligned} f(x, y) &= \mathcal{E}(x^{M_i} y^{U_i}) = [K(y)]^{\lambda N} \mathcal{E} \left[ K(x)^{M_{i-1}} K(y)^{U_{i-1}} \left[ \frac{K(G(x, y))}{K(y)} \right]^{U_{i-1} - \lambda N} \right] \\ &= [K(G(x, y))]^{\lambda N} \mathcal{E} \left[ (K(x))^{M_{i-1}} (K(G(x, y)))^{U_{i-1}} \right]. \end{aligned} \quad (4)$$

Now, substitute  $x_1 = K(x)$ ,  $y_1 = K(G(x, y))$  into equation (4) above. Repeating the above process gives

$$\mathcal{E}(x_1^{M_{i-1}} y_1^{U_{i-1}}) = [K(G(x_1, y_1))]^{\lambda N} \mathcal{E} \left[ (K(x_1))^{M_{i-2}} (K(G(x_1, y_1)))^{U_{i-2}} \right]$$

But,  $G(x_1, y_1) = px_1 + qy_1 = pK(x) + qK(G(x, y)) = K(G_2(x, y))$  as seen above, and

$$\begin{aligned} K(G(x_1, y_1)) &= 1 - \phi + \phi G(x_1, y_1) = 1 - \phi + \phi(pK(x) + qK(G(x, y))) \\ &= 1 - \phi + \phi pK(x) + \phi q(1 - \phi + \phi(px + qy)) \\ &= 1 - \phi^2 + \phi^2 px + \phi^2 q^2 y + \phi^2 qpx \\ &= 1 - \phi^2 + \phi^2 x(1 - q^2) + \phi^2 q^2 y = K_2(G_2(x, y)) \end{aligned}$$

Therefore, substituting into equation (1),

$$\mathcal{E}(x^{M_i} y^{U_i}) = [K(G(x, y))]^{\lambda N} [K_2(G_2(x, y))]^{\lambda N} \mathcal{E}(x_1^{M_{i-2}} y_1^{U_{i-2}}).$$

Using recursion to the second capture occasion, we have

$$\mathcal{E}(x^{M_i} y^{U_i}) = [K(G(x, y)) K_2(G_2(x, y)) \dots K_{i-2}(G_{i-2}(x, y))]^{\lambda N} \mathcal{E}[(K_{i-2}(x))^{M_2} (K_{i-2}(G_{i-2}(x, y)))^{U_2}]$$

At this point we must recognize a difference in the conditional distributions of  $U_2$  and  $M_2$  from what has been used previously. We have  $U_2$  distributed as  $Bin(N - u_1, \phi)$ ;  $M_2$  is distributed as  $Bin(u_1, \phi)$ . Making use of this, we have

$$\begin{aligned}
 \mathcal{E}(\tilde{x}^{M_2} \tilde{y}^{U_2}) &= \mathcal{E}[\mathcal{E}_1(\tilde{x}^{M_2}) \mathcal{E}_1(\tilde{y}^{U_2})] \\
 &= \mathcal{E}[(1 - \phi + \phi \tilde{x})^{u_1} (1 - \phi + \phi \tilde{y})^{N - u_1}] \\
 &= \mathcal{E}[(K(\tilde{x}))^{u_1} (K(\tilde{y}))^{N - u_1}] \\
 &= (K(\tilde{y}))^N \left[ q + p \frac{K(\tilde{x})}{K(\tilde{y})} \right]^N \\
 &= [qK(\tilde{y}) + pK(\tilde{x})]^N
 \end{aligned}$$

where  $\tilde{x} = K_{i-2}(x)$  and  $\tilde{y} = K_{i-2}(G_{i-2}(x, y))$ . Making use of our composition of functions  $K$  and  $G$ , we have

$$\begin{aligned}
 pK(\tilde{x}) + qK(\tilde{y}) &= pK(K_{i-1}(x)) + qK(K_{i-2}(G_{i-2}(x, y))) \\
 &= pK_{i-1}(x) + qK_{i-1}(G_{i-2}(x, y)) = K_{i-1}(G_{i-1}(x, y))
 \end{aligned}$$

so we have  $\mathcal{E}(\tilde{x}^{M_2} \tilde{y}^{U_2}) = [K_{i-1}(G_{i-1}(x, y))]^N$ . Putting everything together, we finally have

$$f(x, y) = \mathcal{E}(x^{M_i} y^{U_i}) = [K_1(G_1(x, y)) K_2(G_2(x, y)) \dots K_{i-2}(G_{i-2}(x, y))]^{N\lambda} (K_{i-1}(G_{i-1}(x, y)))^N.$$

Moments for  $M_i$  and  $U_i$  (see appendix A) can easily be found by taking the natural logarithm of  $f(x, y)$  and differentiating with respect to  $x$  or  $y$ , respectively, then evaluating the derivative at the point  $(1, 1)$ .



In an analogous manner, we can find  $f(x,y) = \mathcal{E}(x^{m_i} y^{u_i})$  using the fact that,

conditional on  $G_i$ ,  $m_i$  is distributed as  $\text{Bin}(M_i, p)$  and  $u_i$  is  $\text{Bin}(U_i + N\lambda, p)$ . We have

$$\begin{aligned} f'(x,y) &= \mathcal{E}(x^{m_i} y^{u_i}) = \mathcal{E}(\mathcal{E}_i(x^{m_i}) \mathcal{E}_i(y^{u_i})) = \mathcal{E}[(q+px)^{M_i} (q+py)^{U_i + N\lambda}] \\ &= (q+py)^{N\lambda} \mathcal{E}(x_1^{M_i} y_1^{U_i}) \\ &= (q+py)^{N\lambda} [K_1(G_1(x_1, y_1)) K_2(G_2(x_1, y_1)) \dots K_{i-2}(G_{i-2}(x_1, y_1))]^{N\lambda} [K_{i-1}(G_{i-1}(x_1, y_1))]^N \end{aligned}$$

where  $x_1 = q+px$  and  $y_1 = q+py$ . Further, we have

$$\begin{aligned} G_j(x_1, y_1) &= (1-q^j)x_1 + q^j y_1 = (1-q^j)(q+px) + q^j(q+py) \\ &= q+p[(1-q^j)x + q^j y] = q+pG_j(x, y). \end{aligned}$$

So, we finally see

$$\begin{aligned} f'(x,y) &= \mathcal{E}(x^{m_i} y^{u_i}) \\ &= (q+py)^{N\lambda} [K_1(q+pG_1(x,y)) \dots K_{i-2}(q+pG_{i-2}(x,y))]^{N\lambda} [K_{i-1}(q+pG_{i-1}(x,y))]^N. \end{aligned}$$

### 3.2 Asymptotic Consistency.

An estimator  $\hat{\theta}$  is said to be strongly consistent if given the true value of the parameter  $\theta$ , we have as  $n \rightarrow \infty$ ,  $\hat{\theta}_n \rightarrow \theta$  almost surely (a.s.) and for  $\epsilon > 0$ , there is an event  $E$  with  $P(E) > 1 - \epsilon$  and an  $n_0$  such that for  $n > n_0$ ,  $\hat{\theta}_n$  satisfies the conditional least squares equations and  $Q_n$  attains a relative minimum at  $\hat{\theta}_n$ . In other words, an estimator

is strongly consistent if it almost surely estimates the proper quantity as the amount of information becomes large. This is generally shown by utilizing a limited (to second derivatives) Taylor expansion of the conditional least squares estimating function  $Q_t(\theta)$  expanded about the "true" value of  $\theta$ . We then demonstrate certain asymptotic behavior of the terms in the Taylor expansion.

Theorem 3.1. Estimates of  $\theta = (p, \phi, N, \lambda)$  obtained from the conditional least squares equation defined in Chapter 2 are strongly consistent.

Proof. In order to establish the desired result we must show (Klimko and Nelson 1978) the following four conditions are satisfied. We are using  $x_i = m_i - u_i$ ,  $\theta^0$  the "true" value of  $\theta$  and  $0 \leq \|\theta - \theta^0\| < \delta$  and taking limits as  $t \rightarrow \infty$ .

$$t^{-1} \frac{\partial^2 Q}{\partial \theta_i^2} \rightarrow 0 \quad a.s.$$

$$t^{-1} \sum_{i=1}^t \left\{ \frac{\partial}{\partial \theta_i} \mathcal{E}(x_i | G_{i-1}) \right\}_{\theta=\theta^0} \left\{ \frac{\partial}{\partial \theta_j} \mathcal{E}(x_i | G_{i-1}) \right\}_{\theta=\theta^0} \rightarrow V_{n \times n} \quad a.s.$$

$$\lim_{t \rightarrow \infty} \sup_{\delta > 0} \left| \left( \frac{\partial^2 Q(\theta)}{\partial \theta_i \partial \theta_j} \right)_{\theta=\theta^0} \right| \leq \infty \quad a.s., \quad 1 \leq i \leq p \quad 1 \leq j \leq p$$

$$t^{-1} \sum_{i=1}^t \left\{ \frac{\partial^2}{\partial \theta_i \partial \theta_j} \mathcal{E}(x_i | G_{i-1}) \right\}_{\theta = \theta^0} \{x_i - \mathcal{E}(x_i | G_{i-1})\} \rightarrow 0 \quad a.s.$$

These will be established by the following series of lemmas which prove Theorem 3.1.

Lemma 3.1.  $\frac{1}{t} \sum_{i=1}^t h_i \rightarrow 0$  a.s. as  $t \rightarrow \infty$ .

Proof. Using a version of the Law of Large Numbers for Martingales (Feller 1968

p. 238), if  $\sum_{i=1}^t i^{-2} \mathcal{E}(h_i^2) < \infty$  for all  $t$ , then we will have  $\frac{1}{t} \sum_{i=1}^t \frac{h_i}{i} \rightarrow 0$  a.s.

Since  $\sum_{i=1}^{\infty} \frac{1}{i^2} < \infty$ , we will have the desired result if  $\mathcal{E}(h_i^2) < \infty$  for all  $i$ . Using conditional expectation, we have

$$\begin{aligned} \mathcal{E}(h_i^2) &= \mathcal{E}(\mathcal{E}_{i-1}(h_i^2)) = \mathcal{E}(\lambda N p(p\phi\psi + q + q\phi) + N_{i-1} p\phi(p\psi + q)) \\ &= \lambda N p(p\phi\psi + q + q\phi) + p\phi(p\psi + q)(N\phi^{i-2} + \lambda N \sum_{k=1}^{i-3} \phi^k) \\ &= \lambda N p(p\phi\psi + q + q\phi) + p\phi(p\psi + q)(N\phi^{i-2} + \lambda N \phi \frac{1 - \phi^{i-3}}{1 - \phi}). \end{aligned}$$

Since we have  $N < \infty$ , and  $p, q, \phi$ , and  $\psi$  are all less than 1, there exists  $B < \infty$  such that

$\mathcal{E}(h_i^2) < B$  for all  $t$ . We therefore have  $\frac{1}{t} \sum_{i=1}^t h_i \rightarrow 0$  a.s. This ends the proof of Lemma

3.1.

Lemma 3.2.  $\frac{1}{t} \sum_{i=1}^t m_i \rightarrow 0$  a.s. as  $t \rightarrow \infty$ .

Proof. Let  $X_t = \sum_{i=1}^t \frac{m_i}{i}$ . Claim  $X_t$  is a submartingale with respect to sigma fields  $\mathcal{F}_t$ .

$$\mathcal{E}_t(X_{t+1}) = \mathcal{E}_t\left(\sum_{i=1}^{t+1} \frac{m_i}{i}\right) = \sum_{i=1}^t \frac{m_i}{i} + E_t\left(\frac{m_{t+1}}{t+1}\right) = X_t + \frac{p\phi(M_t + u_t)}{t+1} \geq X_t.$$

So,  $X_t$  is a submartingale. Further, claim  $X_t$  converges a.s. as  $t \rightarrow \infty$ .

By Stout (1974 p.47), if  $\sum_{i=1}^{\infty} \mathcal{E}\left(\frac{m_i}{i}\right)^k < \infty$  for some  $0 < k \leq 2$ , then  $X_t$  converges a.s.

Consider  $k = 2$ .

We will have  $\sum_{i=1}^t \mathcal{E}\left(\frac{m_i}{i}\right)^2 < \infty$  if  $\mathcal{E}(m_i^2) \leq K < \infty$  for all  $i$  since  $\sum_{i=1}^{\infty} \frac{1}{i^2} < \infty$ . From our probability generating function, we can show (see Appendix A)

$$\begin{aligned} \mathcal{E}(m_i)^2 &= \text{Var}(m_i) + [\mathcal{E}(m_i)]^2 = \left[ Np\phi^{i-1}(1-q^{i-1}) + N\lambda p \left[ \frac{\phi(1-\phi^{i-2})}{1-\phi} - \frac{q\phi(1-(q\phi)^{i-2})}{1-q\phi} \right] \right]^2 \\ &+ p \left[ Np\phi^{i-1}(1-q^{i-1}) + \lambda N \left( \frac{\phi(1-\phi^{i-2})}{1-\phi} - \frac{q\phi(1-(q\phi)^{i-2})}{1-q\phi} \right) \right] \\ &+ p^2 \left[ 2\lambda N\phi^2 q \frac{1-(\phi^2 q)^{i-2}}{1-\phi^2 q} - \lambda N\phi^2 \frac{1-\phi^{2(i-2)}}{1-\phi^2} - N\lambda\phi^2 q^2 \frac{1-(\phi^2 q^2)^{i-2}}{1-\phi^2 q^2} - N\phi^{2(i-2)}(1-q^{i-1})^2 \right]. \end{aligned}$$

$\mathcal{E}(m_i^2)$  is clearly less than  $\infty$  for all  $i$  since  $N < \infty$  and  $0 \leq p, q, \phi \leq 1$ . In fact,

$\sum_{i=1}^{\infty} \mathcal{E}\left(\frac{m_i}{i}\right)^k$  converges for all  $k > 1$ . From the Kronecker lemma, we therefore have

$\frac{1}{t} \sum_{i=1}^t m_i$  converges a.s. to 0 as  $t \rightarrow \infty$ . This ends the proof of Lemma 3.2.

Corollary 3.1.  $\frac{1}{t} \sum_{i=1}^{\infty} u_i m_i$  converges a.s. as  $t \rightarrow \infty$ .

Proof. We use the same reasoning as in the lemma above. Observe that from the probability generating function for  $u_i$  and  $m_i$  all moments are bounded for all  $i$ .

Corollary 3.2.  $\frac{1}{t} \sum_{i=1}^t m_i^2$  converges a.s. as  $t \rightarrow \infty$ .

Proof. Again, we use the fact that all moments for  $m_i$  are bounded, and the same reasoning as in the lemma.

Corollary 3.3. There exists  $k < \infty$  such that  $m_i < k$  for all  $i$ .

Proof. Assume not. Then, given  $\delta > 0$ ,  $\exists n$  such that  $\frac{1}{n} \sum_{i=1}^n m_i > \delta$ . This implies we cannot have  $\frac{1}{t} \sum_{i=1}^t m_i \rightarrow 0$  a.s. Therefore,  $m_i$  are bounded.

Lemma 3.3.  $\frac{1}{t} \frac{\partial Q_t}{\partial \theta_i} \rightarrow 0$  a.s. as  $t \rightarrow \infty$ , and  $\theta = (p, \phi, N, \lambda)'$

Proof.

$$\frac{1}{t} \frac{\partial Q_t}{\partial N} = \frac{2}{t} \left( p(-u_1 + Np) + \sum_{i=2}^t h_i(\lambda p) \right) = \frac{2p(-u_1 + Np)}{t} + \frac{2\lambda p}{t} \sum_{i=2}^t h_i$$

By the lemma on convergence of  $\frac{1}{t} \sum h_i$  (Lemma 3.1), this converges a.s. to 0.

$\frac{1}{t} \frac{\partial Q_t}{\partial \lambda} = \frac{2pN}{t} \sum_{i=2}^t h_i$  also converges a.s. to 0 by the same reasoning.

$$\frac{1}{t} \frac{\partial Q_t}{\partial p} = \frac{2N(-u_1 + Np)}{t} + \frac{2}{t} \sum_{i=2}^t h_i(2\phi u_{i-1} + \lambda N). \text{ Clearly, the first term converges to 0 as}$$

$t \rightarrow \infty$ . As above, the term  $\frac{2\lambda N}{t} \sum_{i=2}^t h_i$  converges a.s. to 0 as  $t \rightarrow \infty$ . For the remaining term, we observe  $\sum_{i=2}^t u_{i-1} \leq N + (t-2)\lambda N$ , so there exists  $K < \infty$  such that  $u_i < K$  for all  $i$ . This implies convergence of the remaining term.

$$\frac{1}{t} \frac{\partial Q_t}{\partial \phi} = \frac{2}{t} \sum_{i=2}^t h_i((1-2p)u_{i-1} - m_{i-1}) = \frac{2(1-2p)}{t} \sum_{i=2}^t h_i u_i - \frac{2}{t} \sum_{i=2}^t h_i m_i. \text{ We have already}$$

shown convergence of the first term. Convergence of the second is obtained using the corollary on boundedness of  $m_i$  (Corollary 3.3 to Lemma 3.2). This concludes the proof of Lemma 3.3.

**Lemma 3.4.**  $\lim_{t \rightarrow \infty} \sup_{\delta > 0} \frac{1}{t\delta} |T_t(\theta^\cdot)_{ij}| < \infty, 1 \leq i, j \leq 4$  where

$$T_t(\theta)_{ij} = \frac{\partial^2 Q_t(\theta^\cdot)}{\partial \theta_i \partial \theta_j} - \frac{\partial^2 Q_t(\theta^0)}{\partial \theta_i \partial \theta_j} \text{ and } \theta^0 \text{ is the "true" value of } \theta \text{ and } 0 \leq |\theta^\cdot - \theta^0| \leq \delta.$$

Proof.

We have  $\frac{1}{t\delta} |T_t(\theta)_{ij}| = \frac{1}{t\delta} \left| \frac{\partial^2 Q_t(\theta)}{\partial \theta_i \partial \theta_j} - \frac{\partial^2 Q_t(\theta^0)}{\partial \theta_i \partial \theta_j} \right| = \frac{1}{t\delta} |f(\theta) - f(\theta^0)|$ . Using the

Mean Value Theorem for multi-variate equations, we have this equal to

$$\frac{1}{t\delta} \left| (N^* - N^0) \frac{\partial f(\theta^{**})}{\partial N} + (p^* - p^0) \frac{\partial f(\theta^{**})}{\partial p} + (\phi^* - \phi^0) \frac{\partial f(\theta^{**})}{\partial \phi} + (\lambda^* - \lambda^0) \frac{\partial f(\theta^{**})}{\partial \lambda} \right| \text{ where}$$

$$\|\theta^0 - \theta^{**}\| \leq \|\theta^0 - \theta^*\| \text{ since } \|\theta^0 - \theta^*\| < \delta \text{ implies } \|N^0 - N^*\| < \delta; \|p^0 - p^*\| < \delta;$$

$$\|\phi^0 - \phi^*\| < \delta; \text{ and } \|\lambda^0 - \lambda^*\| < \delta. \text{ So, we have}$$

$$\frac{1}{t\delta} |T_t(\theta)_{ij}| < \frac{1}{t} \left| \frac{\partial f(\theta^{**})}{\partial N} + \frac{\partial f(\theta^{**})}{\partial p} + \frac{\partial f(\theta^{**})}{\partial \phi} + \frac{\partial f(\theta^{**})}{\partial \lambda} \right| = A. \text{ We must show A finite}$$

for all values of t. In other words, we must show all third partial derivatives of  $Q_t(\theta)$  are

bounded.

There are ten second partial derivatives to evaluate and then take a partial

derivatives for each variable.

$$1. f = \frac{\partial^2 Q_t(\theta)}{\partial N^2} = 2p^2 + 2\lambda^2 p^2(t-1). \text{ Partial derivatives with respect to N and } \phi \text{ are}$$

$$\text{zero. We have } \frac{1}{t\delta} |T_t(\theta)_{ij}| < \frac{1}{t} (4\lambda^2 p(t-1) + 4p + 4\lambda p^2(t-1)) = 4\lambda^2 p + 4\lambda p^2 < \infty.$$

$$2. f = \frac{\partial^2 Q_t(\theta)}{\partial N \partial p} = (4Np - 2u_1) + 2\lambda \sum_{i=2}^t (m_i - u_i - \phi m_{i-1} + (1-2p)\phi u_{i-1} + \lambda Np) + \\ + 2\lambda p \sum_{i=2}^t (\lambda N - 2\phi u_{i-1}).$$

We have the following partial derivatives.

$$\frac{\partial f}{\partial N} = 4p + 4\lambda^2 p(t-1),$$

$$\frac{\partial f}{\partial p} = 4N + 4\lambda \sum_{i=2}^t (\lambda N - 2\phi u_{i-1}),$$

$$\frac{\partial f}{\partial \phi} = 2\lambda \sum_{i=2}^t ((1-2p)u_{i-1} - m_{i-1}) - 4\lambda p \sum_{i=2}^t u_{i-1}$$

$$\frac{\partial f}{\partial \lambda} = 2 \sum_{i=2}^t h_i + 2p \sum_{i=2}^t (\lambda N - 2\phi u_{i-1}) + 4\lambda Np(t-1).$$

So,

$$\begin{aligned} \frac{1}{t\delta} |T_t(\theta')_{ij}| &< \frac{1}{t} |4p + 4\lambda^2 p(t-1) + 4N + 4\lambda^2 N(t-1) - 8\lambda\phi \sum_{i=2}^t u_{i-1} + 2\lambda(1-2p) \sum_{i=2}^t u_{i-1} \\ &\quad - 2\lambda \sum_{i=2}^t m_{i-1} - 4\lambda p \sum_{i=2}^t u_{i-1} + 2 \sum_{i=2}^t h_i + 2p\lambda N(t-1) - 4p\phi \sum_{i=2}^t u_{i-1} + 4\lambda Np(t-1)| \end{aligned}$$

Using lemmas 3.1 and 3.2, we have this quantity converging almost surely to a quantity

less than  $4\lambda^2 p + 4\lambda^2 N + 8\lambda^2 \phi N + 2\lambda^2 N(1-2p) + 4\lambda^2 Np + 2\lambda Np + 4\lambda Np\phi$  as  $t \rightarrow \infty$ .

3.  $f = \frac{\partial^2 Q_t(\theta)}{\partial N \partial \phi} = 2\lambda p \sum_{i=2}^t ((1-2p)u_{i-1} - m_{i-1})$ . The partial derivatives with respect to  $N$

and  $\phi$  are zero. We have  $\frac{\partial f}{\partial p} = 2\lambda \sum_{i=2}^t ((1-2p)u_{i-1} - m_{i-1}) - 4\lambda p \sum_{i=2}^t u_{i-1}$  and

$\frac{\partial f}{\partial \lambda} = 2p \sum_{i=2}^t ((1-2p)u_{i-1} - m_{i-1})$ . Therefore,

$$\frac{1}{t\delta} |T_t(\theta')_{ij}| < \frac{1}{t} [2\lambda(1-2p) + 4\lambda p + 2p(1-2p)] \sum_{i=2}^t u_{i-1} + 2(\lambda + p) \sum_{i=2}^t m_{i-1}.$$

Again using the boundedness of the  $m_i$  (corollary 3.3 to lemma 3.2) we have this converging almost

surely to a quantity less than  $2\lambda^2 |1-2p| N + 4\lambda^2 pN + 2p\lambda N |1-2p| < \infty$ .

4.  $f = \frac{\partial^2 Q}{\partial N \partial \lambda} = 2p \sum_{i=2}^t h_i + 2\lambda p^2 N(t-1)$ . We have the following partial derivatives of  $f$ .

$$\frac{\partial f}{\partial N} = 4p^2 \lambda (t-1) \qquad \frac{\partial f}{\partial \phi} = 2p \sum_{i=2}^t ((1-2p)u_{i-1} - m_{i-1})$$

$$\frac{\partial f}{\partial p} = 2 \sum_{i=2}^t h_i + 2p \sum_{i=2}^t (\lambda N - 2\phi u_{i-1}) + 4\lambda p N(t-1) \qquad \frac{\partial f}{\partial \lambda} = 4p^2 N(t-1)$$

Therefore,



$$\begin{aligned} \frac{1}{\delta t} |T_t(\theta^*)_{ij}| &< \frac{1}{t} |4p^2\lambda(t-1) + 2p(1-2p)N + 2p(1-2p)\lambda N(t-1) + 2p\sum_{i=1}^t m_i + 2\sum_{i=2}^t h_i \\ &+ 2p\lambda N(t-1) + 4p\phi N + 4p\phi\lambda N(t-1) + 4\lambda p N(t-1) + 4p^2N(t-1)| \end{aligned}$$

which will converge almost surely to  $4p^2\lambda + 2p|1-2p|\lambda N + 6p\lambda N + 4p\phi\lambda N + 4p^2N < \infty$ .

5.  $f = \frac{\partial^2 Q}{\partial p^2} = 2N^2 + 2\sum_{i=2}^t (\lambda N - 2\phi u_{i-1})^2$ . We have the following partial derivatives of  $f$ .

$$\frac{\partial f}{\partial N} = 4N + 4\lambda \sum_{i=2}^t (\lambda N - 2\phi u_{i-1}) \quad \frac{\partial f}{\partial p} = 0$$

$$\frac{\partial f}{\partial \phi} = 8\sum_{i=2}^t (\lambda N - 2\phi u_{i-1})(-2u_{i-1}) \quad \frac{\partial f}{\partial \lambda} = 4N\sum_{i=2}^t (\lambda N - 2\phi u_{i-1})$$

We have

$$\begin{aligned} \frac{1}{\delta t} |T_t(\theta^*)_{ij}| &< \frac{1}{t} |2\lambda(1-2p)N + 2\lambda^2(1-2p)N(t-1) + (2\lambda + 2N)\sum_{i=1}^t m_{i-1} + 4\lambda p N \\ &+ 4\lambda^2 p N(t-1) + 8\lambda^2 N^2 + 8\lambda^2 N^2(t-1) + \phi N^2 + 16\phi\lambda^2 N^2(t-1) + 8(1-2p)N^2 \\ &+ 8(1-2p)\lambda^2 N^2(t-1) + 8\sum_{i=2}^t m_i u_i + 2(1-2p)N^2 + 2(1-2p)\lambda N^2(t-1) + 2N^2 t \\ &+ 2\lambda N^2 p(t-1) \end{aligned}$$

which converges a.s. to the quantity (using lemma 3.2 and its corollary 3.1)

$$2\lambda^2|1-2p| + 4\lambda^2 p N + 8\lambda^2 N^2 + 16\phi\lambda^2 N^2 + 8|1-2p|\lambda^2 N^2 + 2|1-2p|\lambda N^2 + 2\lambda N^2 p < \infty.$$

6.  $f = \frac{\partial^2 Q}{\partial p \partial \phi} = 2\sum_{i=2}^t [((1-2p)u_{i-1} - m_{i-1})(\lambda N - 2\phi u_{i-1}) - 2u_{i-1}h_i]$  Which has the following partial derivatives.

$$\frac{\partial f}{\partial N} = 2\sum_{i=2}^t [((1-2p)u_{i-1} - m_{i-1})\lambda - 2u_{i-1}\lambda p] \quad \frac{\partial f}{\partial p} = 4\sum_{i=2}^t [(\lambda N - 2\phi u_{i-1})(-2u_{i-1})]$$

$$\begin{aligned}\frac{\partial f}{\partial \phi} &= -8 \sum_{i=2}^t ((1-2p)u_{i-1} - m_{i-1})u_{i-1} \\ \frac{\partial F}{\partial \lambda} &= 2 \sum_{i=1}^t [((1-2p)u_{i-1} - m_{i-1})N - 2u_{i-1}Np]\end{aligned}$$

So we have

$$\begin{aligned}\frac{1}{t\delta} |T_t(\theta^*)_{ij}| &< \frac{1}{t} [|1-2p| (2\lambda N + 2\lambda^2 N(t-1) + 8N^2 + 8\lambda^2 N^2(t-1) + 2N^2 + 2\lambda N^2(t-1)) \\ &\quad + (2\lambda + 2N) \sum_{i=2}^t m_i + 4\lambda p N + 4\lambda^2 p N(t-1) + 8\lambda N^2 + 8\lambda^2 N^2(t-1) + 16\phi N^2 \\ &\quad + 16\phi \lambda^2 N^2(t-1) + 8 \sum_{i=2}^t m_i u_i + 2N^2 p + 2\lambda N^2 p(t-1)]\end{aligned}$$

which converges a.s. to  $|1-2p| (2\lambda^2 N + 8\lambda^2 N^2 + 2\lambda N^2) + 2\lambda N(2\lambda p + 4\lambda N + 8\phi \lambda N + 2Np)$ .

$$7. f = \frac{\partial^2 Q}{\partial p \partial \lambda} = 2 \sum_{i=1}^t [Np(\lambda N - 2\phi u_{i-1} + h_i N)].$$

$$\begin{aligned}\frac{\partial f}{\partial N} &= 2 \sum_{i=2}^t [2Np\lambda - 2p\phi u_{i-1} + \lambda Np + h_i] & \frac{\partial f}{\partial \lambda} &= 4N^2 p(t-1) \\ \frac{\partial f}{\partial \phi} &= 2 \sum_{i=2}^t [-2Np u_{i-1} + N((1-2p)u_{i-1} - m_{i-1})] & \frac{\partial f}{\partial p} &= 4 \sum_{i=2}^t [N^2 \lambda - 2\phi N u_{i-1}]\end{aligned}$$

$$\begin{aligned}\frac{1}{t\delta} |T_t(\theta^*)_{ij}| &< \frac{1}{t} [(t-1)(4Np\lambda + 4Np\phi\lambda + 2Np\lambda + 4N^2 p + 4N^2 \lambda + 8\phi \lambda N + 4\lambda N^2 p) \\ &\quad + 4p\phi N + \sum_{i=2}^t h_i + 8\phi N^2 + 4N^2 p + 2N^2 |1-2p| (1 + \lambda(t-1)) + 2N \sum_{i=2}^t m_i]\end{aligned}$$

which converges a.s. to  $6Np\lambda + 4Np\phi\lambda + 4N^2 \lambda + 8\phi \lambda N + 4\lambda N^2 p + 2\lambda N^2 |1-2p|$ .

$$8. f = \frac{\partial^2 q}{\partial \phi^2} = 2 \sum_{i=2}^t ((1-2p)u_{i-1} - m_{i-1})^2.$$

The only non-zero partial derivative of this is with respect to p.

$$\frac{\partial f}{\partial p} = 4 \sum_{i=2}^t ((1-2p)u_{i-1} - m_{i-1})(-2u_{i-1}).$$

So therefore,

$$\frac{1}{t\delta} |T_t(\theta')_{ij}| < \frac{1}{t} [8N^2 |1-2p| (1+\lambda^2(t-1)) + 8 \sum_{i=2}^t u_{i-1} m_{i-1}] - 8 |1-2p| N^2 \lambda^2 < \infty$$

9.  $f = \frac{\partial^2 Q}{\partial \phi \partial \lambda} = 2Np \sum_{i=2}^t ((1-2p)u_{i-1} - m_{i-1})$ . Partial derivatives with respect to  $\phi$  and  $\lambda$  are zero.

$$\frac{\partial f}{\partial N} = 2p \sum_{i=2}^t ((1-2p)u_{i-1} - m_{i-1}) \quad \frac{\partial f}{\partial p} = 2N \sum_{i=2}^t ((1-2p)u_{i-1} - m_{i-1}) - 4Np \sum_{i=2}^t u_{i-1}.$$

Therefore,

$$\begin{aligned} \frac{1}{t\delta} |T_t(\theta')_{ij}| < \frac{1}{t} [ |1-2p| (2pN + 2p\lambda N(t-1) + 2N^2 + 2\lambda N^2(t-1)) + (2p+2N) \sum_{i=2}^t m_i \\ + 4N^2 p + 4\lambda N^2 p(t-1) ] \end{aligned}$$

which converges a.s. to  $2\lambda N |1-2p| (p+N) + 4\lambda N^2 p < \infty$

10.  $f = \frac{\partial^2 Q}{\partial \lambda^2} = 2N^2 p^2 (t-1)$ . Partial derivatives with respect to  $\phi$  and  $\lambda$  are zero, and

we have  $\frac{\partial f}{\partial N} = 4Np^2(t-1)$  and  $\frac{\partial f}{\partial p} = 4N^2 p(t-1)$ . Therefore,

$$\frac{1}{t\delta} |T_t(\theta')_{ij}| < \frac{1}{t} [4Np(t-1)(p+N)] - 4Np(p+N) < \infty.$$

This completes this proof of Lemma 3.4.

Lemma 3.5.  $\frac{1}{t} \frac{\partial Q_t}{\partial \theta_i} \rightarrow 0$  a.s. as  $t \rightarrow \infty$ , and  $\theta = (p, \phi, N, \lambda)'$

Proof.

$$\frac{1}{t} \frac{\partial Q_t}{\partial N} = \frac{2}{t} \left( p(-u_1 + Np) + \sum_{i=2}^t h_i(\lambda p) \right) = \frac{2p(-u_1 + Np)}{t} + \frac{2\lambda p}{t} \sum_{i=2}^t h_i$$

By the lemma on convergence of  $\frac{1}{t} \sum h_i$  (Lemma 3.1), this converges a.s. to 0.

$\frac{1}{t} \frac{\partial Q_t}{\partial \lambda} = \frac{2pN}{t} \sum_{i=2}^t h_i$  also converges a.s. to 0 by the same reasoning.

$\frac{1}{t} \frac{\partial Q_t}{\partial p} = \frac{2N(-u_1 + Np)}{t} + \frac{2}{t} \sum_{i=2}^t h_i(2\phi u_{i-1} + \lambda N)$ . Clearly, the first term converges to 0 as

$t \rightarrow \infty$ . As above, the term  $\frac{2\lambda N}{t} \sum_{i=2}^t h_i$  converges a.s. to 0 as  $t \rightarrow \infty$ . For the remaining term,

we observe  $\sum_{i=2}^t u_{i-1} \leq N + (t-2)\lambda N$ , so there exists  $K < \infty$  such that  $u_i < K$  for all  $i$ . This

implies convergence of the remaining term.

$$\frac{1}{t} \frac{\partial Q_t}{\partial \phi} = \frac{2}{t} \sum_{i=2}^t h_i((1-2p)u_{i-1} - m_{i-1}) = \frac{2(1-2p)}{t} \sum_{i=2}^t h_i u_i - \frac{2}{t} \sum_{i=2}^t h_i m_i$$

We have already shown convergence of the first term. Convergence of the second is obtained using the

corollary on boundedness of  $m_i$  (Corollary 3.3 of Lemma 3.2). This completes the proof

of Lemma 3.5.

**Lemma 3.6.**  $\frac{1}{t} \sum_{i=1}^t h_i \left( \frac{\partial^2}{\partial \theta_i \partial \theta_j} \mathcal{E}_{\theta^0}((m_i - u_i) | G_{i-1}) \right) \rightarrow 0$  a.s. as  $t \rightarrow \infty$ , for

$1 \leq i, j \leq 4$ .

**Proof.** We have  $\mathcal{E}_{\theta^0}((m_i - u_i) | G_{i-1}) = \phi m_{i-1} - (1 - 2p)\phi u_{i-1} - \lambda N p$ . There are ten second partial derivatives to examine.

1.  $\frac{\partial^2}{\partial N^2} \mathcal{E}_{\theta^0}((m_i - u_i) | G_{i-1}) = 0$ . Therefore the sum converges to 0 trivially.

2.  $\frac{\partial^2}{\partial N \partial \phi} \mathcal{E}_{\theta^0}((m_i - u_i) | G_{i-1}) = 0$ .

3.  $\frac{\partial^2}{\partial N \partial p} \mathcal{E}_{\theta^0}((m_i - u_i) | G_{i-1}) = -\lambda$ . Therefore, we have  $\frac{-\lambda}{t} \sum_{i=1}^t h_i$  which converges a.s. to

zero by lemma 3.1.

4.  $\frac{\partial^2}{\partial N \partial \lambda} \mathcal{E}_{\theta^0}((m_i - u_i) | G_{i-1}) = -p$ . Converges by the same reasoning as in 3.

5.  $\frac{\partial^2}{\partial p^2} \mathcal{E}_{\theta^0}((m_i - u_i) | G_{i-1}) = 0$

6.  $\frac{\partial^2}{\partial p \partial \phi} \mathcal{E}_{\theta^0}((m_i - u_i) | G_{i-1}) = 2u_{i-1}$ . The quantity  $\frac{1}{t} \sum_{i=1}^t h_i u_{i-1}$  has been shown to

converge a.s. to 0 in lemma 3.5.

7.  $\frac{\partial^2}{\partial p \partial \lambda} \mathcal{E}_{\theta^0}((m_i - u_i) | G_{i-1}) = -N$ . We have a.s. convergence by the same reasoning as in 3.

8.  $\frac{\partial^2}{\partial \phi^2} \mathcal{E}_{\theta^0}((m_i - u_i) | G_{i-1}) = 0$ . Convergence is trivial.

9.  $\frac{\partial^2}{\partial \phi \partial \lambda} \mathcal{E}_{\theta^0}((m_i - u_i) | G_{i-1}) = 0$

10.  $\frac{\partial^2}{\partial \lambda^2} \mathcal{E}_{\theta^0}((m_i - u_i) | G_{i-1}) = 0$

This completes the proof of Lemma 3.6.

Lemma 3.7.  $\frac{1}{t} \sum_{i=1}^t \left\{ \frac{\partial}{\partial \theta_i} \mathcal{E}_{\theta^0}(m_i - u_i | G_{i-1}) \right\} \left\{ \frac{\partial}{\partial \theta_j} \mathcal{E}_{\theta^0}(m_i - u_i | G_{i-1}) \right\} - V$  a.s. as  $t \rightarrow \infty$ ,  $1 \leq i, j \leq 4$ .

Proof. There are ten products of partial derivatives to check. First, we have

$\mathcal{E}_{\theta^0}(m_i - u_i | G_{i-1}) = \phi m_{i-1} - (1 - 2p) \phi u_{i-1} - \lambda N p$ ,  $i \geq 2$ .  $\mathcal{E}_{\theta^0}(m_1 - u_1) = -N p$ . Clearly, we

have  $\frac{1}{t} \left( \frac{\partial}{\partial \theta_i} \right) \left( \frac{\partial}{\partial \theta_j} \right) (-N p) = 0$  as  $t \rightarrow \infty$ , so we will ignore this term in the following.

$$1. \frac{1}{t} \sum_{i=2}^t \left( \frac{\partial}{\partial N} \right) \left( \frac{\partial}{\partial N} \right) = \frac{1}{t} \sum_{i=2}^t \lambda^2 p^2 = \lambda^2 p^2$$

$$2. \frac{1}{t} \left( \frac{\partial}{\partial N} \right) \left( \frac{\partial}{\partial p} \right) = \frac{1}{t} \sum_{i=2}^t (-\lambda p)(2\phi u_{i-1} - \lambda N) = \frac{\lambda^2 N p}{t} \sum_{i=2}^t 1 - \frac{2\lambda p \phi}{t} \sum_{i=2}^t u_{i-1} = A.$$

We have  $\frac{\lambda^2 N p}{t} \sum_{i=2}^t 1 \leq A \leq \frac{\lambda^2 N p}{t} \sum_{i=2}^t 1 + \frac{2\lambda p \phi}{t} (N + (t-2)\lambda N)$ , since

$$\sum_{i=1}^{t-1} u_i \leq N + (t-2)\lambda N \text{ which is achieved if } p = 1. \text{ Therefore, as } t \rightarrow \infty, \text{ we have}$$

$$\lambda^2 N p \leq \lim_{t \rightarrow \infty} A \leq \lambda^2 N p (1 - 2p).$$

$$3. \frac{1}{t} \sum_{i=2}^t \left( \frac{\partial}{\partial N} \right) \left( \frac{\partial}{\partial \lambda} \right) = \frac{1}{t} \sum_{i=2}^t \lambda N p^2 - \lambda N p^2.$$

$$4. \frac{1}{t} \sum_{i=2}^t \left( \frac{\partial}{\partial \lambda} \right) \left( \frac{\partial}{\partial \lambda} \right) = \frac{1}{t} \sum_{i=2}^t N^2 p^2 - N^2 p^2.$$

$$5. \frac{1}{t} \sum_{i=2}^t \left( \frac{\partial}{\partial \lambda} \right) \left( \frac{\partial}{\partial p} \right) = \frac{1}{t} \sum_{i=2}^t (-N p)(2\phi u_{i-1} - \lambda N) = \frac{N^2 \lambda p}{t} \sum_{i=2}^t 1 - \frac{2N p \phi}{t} \sum_{i=2}^t u_{i-1} = A.$$

Again, as in 3 above, we have as  $t \rightarrow \infty$ ,  $N^2 \lambda p \leq \lim_{t \rightarrow \infty} A \leq N^2 \lambda p (1 + 2\phi)$ .

$$6. \frac{1}{t} \sum_{i=2}^t \left( \frac{\partial}{\partial N} \right) \left( \frac{\partial}{\partial \phi} \right) = \frac{\lambda p}{t} \sum_{i=2}^t ((1-2p)u_{i-1} - m_{i-1}). \text{ Using the results of lemma 3.2, we}$$

have

$$\frac{\lambda p}{t} \sum_{i=2}^t ((1-2p)u_{i-1} - m_{i-1}) \leq \frac{\lambda p(1-2p)}{t} [N + (t-2)\lambda N] + \frac{\lambda p K}{t} \sum 1 - \lambda^2 N p (1-2p) + \lambda p K$$

as  $t \rightarrow \infty$ .

$$7. \frac{1}{t} \sum_{i=2}^t \left( \frac{\partial}{\partial \lambda} \right) \left( \frac{\partial}{\partial \phi} \right) = \frac{Np}{t} \sum_{i=2}^t ((1-2p)u_{i-1} - m_{i-1}). \text{ As in 6 above, this approaches a}$$

limit less than  $N^2\lambda p(1-2p) + NpK$  as  $t \rightarrow \infty$ .

$$8. \frac{1}{t} \sum_{i=2}^t \left( \frac{\partial}{\partial p} \right) \left( \frac{\partial}{\partial p} \right) = \frac{1}{t} \sum_{i=2}^t (2\phi u_{i-1} - \lambda N)^2 = \frac{4\phi^2}{t} \sum_{i=2}^t u_{i-1}^2 - \frac{4\phi\lambda N}{t} \sum_{i=2}^t u_{i-1} + \frac{\lambda^2 N^2}{t} \sum_{i=2}^t 1.$$

This will converge if the first term converges. As before,  $u_i$  is maximized if  $p = 1$ , so

similarly  $u_i^2$  is maximized if  $p = 1$ . Therefore, we have  $\sum_{i=1}^t u_i^2 \leq N^2 + \lambda^2 N^2(t-1)$ .

Therefore, we approach a limit less than  $\lambda^2 N^2(4\phi^2 + 4\phi + 1)$  as  $t \rightarrow \infty$ .

$$9. \frac{1}{t} \sum_{i=2}^t \left( \frac{\partial}{\partial p} \right) \left( \frac{\partial}{\partial \phi} \right) = \frac{1}{t} \sum_{i=2}^t (2\phi m_{i-1} u_{i-1} - \lambda N m_{i-1} - 2(1-2p)\phi u_{i-1}^2 + \lambda N(1-2p)u_{i-1}). \text{ All}$$

terms except the first have already been shown to be convergent. The first term converges

by corollary 3.3 to lemma 3.2 ( $m_i$  are bounded). Therefore, we have this quantity

approaching a limit less than  $\lambda N(2\phi K + K + 2(1-2p)\phi + \lambda N(1-2p))$  as  $t \rightarrow \infty$ .

$$10. \frac{1}{t} \left( \frac{\partial}{\partial \phi} \right) \left( \frac{\partial}{\partial \phi} \right) = \frac{1}{t} \sum_{i=2}^t (m_{i-1}^2 - 2(1-2p)u_{i-1}m_{i-1} + (1-2p)^2 u_{i-1}^2). \text{ Using the results of}$$

corollaries 3.1 and 3.2 to lemma 3.2, we have all terms convergent, with a limit less than

$K^2 + 2(1-2p)\lambda N + (1-2p)^2 \lambda N$  as  $t \rightarrow \infty$ .



### 3.3 Asymptotic Normality of Estimates.

If the parameter estimates obtained from the conditional least squares function are asymptotically normal, confidence intervals for the true values can be formed using normal quantiles. We first establish a result which amounts to the conditional variance of the quantity  $m_i - u_i$  which is needed in the following theorem.

#### Lemma 3.8.

$\mathcal{E}_{i-1}^{\cdot}(h_i^2) = p\phi(q + p\psi)N_{i-1} + 4p^2\phi^2qU_{i-1} + \lambda Np(q + q\phi + p\phi\psi + 4p^2\phi^2q)$ , where

$$\mathcal{E}_{i-1}^{\cdot}(\bullet) = \mathcal{E}(\bullet | G_{i-1}).$$

Proof.

$$\mathcal{E}_{i-1}^{\cdot}(h_i^2) = \mathcal{E}_{i-1}^{\cdot}(m_i - u_i - \mathcal{E}_{i-1}^{\cdot}(m_i - u_i))^2 = \text{Var}_{i-1}^{\cdot}(m_i - u_i). \text{ We use}$$

conditional variances to find

$$\text{Var}_{i-1}^{\cdot}(m_i - u_i) = \mathcal{E}_{i-1}^{\cdot}(\text{Var}_{i-1}(m_i - u_i)) + \text{Var}_{i-1}^{\cdot}(\mathcal{E}_{i-1}^{\cdot}(m_i - u_i)) \text{ where}$$

$$\mathcal{E}_{i-1}^{\cdot}(\bullet) = \mathcal{E}(\bullet | \mathcal{J}_{i-1}).$$

Since  $m_i$  and  $u_i$  are conditionally independent with regard to the sigma field  $\mathcal{J}_{i-1}$ ,  $\text{Var}_{i-1}(m_i - u_i) = \text{Var}_{i-1}(m_i + u_i) = \text{Var}_{i-1}(n_i)$ . Again using conditional variances,  $\text{Var}_{i-1}(n_i) = \mathcal{E}_{i-1}^{\cdot}(\text{Var}_i^{\cdot}(n_i)) + \text{Var}_{i-1}^{\cdot}(\mathcal{E}_i^{\cdot}(n_i))$ . But

$$\text{Var}_i^{\cdot}(n_i) = pq(N_i + \lambda N) \text{ and } \mathcal{E}_{i-1}^{\cdot}(\text{Var}_i^{\cdot}(n_i)) = pq\lambda N + pq\phi\lambda N + pq\phi N_{i-1}.$$

Further,  $\mathcal{E}_i^{\cdot}(n_i) = p(N_i + \lambda N)$  and  $\text{Var}_{i-1}^{\cdot}(\mathcal{E}_i^{\cdot}(n_i)) = p^2\phi\psi(N_{i-1} + \lambda N)$ . So,

$Var_{i-1}(n_i) = p\phi(q+p\psi)N_{i-1} + \lambda Np(q+q\phi+p\phi\psi)$ . All quantities in this variance are constants with respect to sigma field  $G_{i-1}$ .

$\mathcal{E}_{i-1}(m_i - u_i) = p\phi(M_{i-1} + u_{i-1}) - p\phi(U_{i-1} + \lambda N - u_{i-1}) - \lambda Np$ . With respect to the sigma field  $G_{i-1}$ , all quantities except  $u_{i-1}$  are constants. Thus, we are left with

$$Var_{i-1}(\mathcal{E}_{i-1}(m_i - u_i)) = Var_{i-1}(2p\phi u_{i-1}) = 4p^2\phi^2 Var_{i-1}(u_{i-1}) = 4p^3\phi^2 q(U_{i-1} + \lambda N).$$

Adding the two results completes the proof of lemma 3.8.

**Theorem 3.2.** Estimates of parameters obtained through our conditional least squares estimating equation are jointly asymptotically normal.

Proof.

Since we are in a multiple parameter setting, we will employ the Cramér-Wold device (Billingsley 1968), showing an arbitrary linear combination of the first partial derivatives approaches normality in the limit, as in Hall and Heyde (1980). In order to show joint asymptotic normality of our estimates, it is necessary to show (Brown 1971) the following two conditions are met:

$$1. \quad \mathcal{E} \left| \frac{V_t^2 - s_t^2}{s_t^2} \right| \rightarrow 0$$

$$2. \frac{1}{s_t^2} \sum_{i=1}^t \mathcal{E}[(X_i)^2 I(|X_i| > \epsilon s_t)] \rightarrow_p 0$$

In the above equations, we have  $X_i = \sum_{j=1}^4 c_j \frac{\partial Q(t)}{\partial \theta_j}$ ;  $V_t^2 = \sum_{i=1}^t \text{Var}(X_i | G_{i-1})$ , a conditional variance; and  $s_t^2 = \mathcal{E}(V_t^2)$ . The second condition is a conditional version of Lindeberg asymptotic negligibility.

We must first obtain  $V_t^2 = \text{Var}_{i-1}(X_i) = \sum_{i=1}^t \text{Var}_{i-1}(h_i) \left( \sum_{j=1}^4 c_j \frac{\partial h_i}{\partial \theta_j} \right)^2$ , since all partial derivatives are measurable with respect to  $G_{i-1}$ . In lemma 3.8 we showed

$$\mathcal{E}_{i-1}(h_i^2) = \begin{cases} pqN & i = 1 \\ p\phi(q + p\psi)N_{i-1} + 4p^3\phi^2qU_{i-1} + \lambda Np(\phi\psi p + q\phi + q + 4p^3\phi^2q) & i > 1 \end{cases}$$

which is  $\text{Var}_{i-1}(h_i)$  since  $\mathcal{E}(h_i | G_{i-1}) = 0$ . Keeping in mind the following partial derivatives

$$\frac{\partial h_i}{\partial p} = \lambda N - 2\phi u_{i-1} \quad \frac{\partial h_i}{\partial \phi} = (1 - 2p)u_{i-1} - m_{i-1} \quad \frac{\partial h_i}{\partial N} = \lambda p \quad \frac{\partial h_i}{\partial \lambda} = Np$$

and that  $N_{i-1} = M_{i-1} + U_{i-1}$ , we have

$$V_t^2 = (c_1 N + c_3 p)^2 p q N + \sum_{i=2}^t (A + B u_{i-1} + C m_{i-1})^2 (D M_{i-1} + F U_{i-1} + K)$$

where

$$A = c_1 \lambda N + c_3 \lambda p + c_4 N p$$

$$B = c_2 (1 - 2p) - 2c_1 \phi$$

$$C = -c_4$$

$$D = p\phi(q + p\psi)$$

$$F = p\phi(q + p\psi) + 4p^3 \phi^2 q$$

$$K = \lambda N p (q + q\phi + p\phi\psi + 4p^3 \phi^2 q).$$

To show condition 1 is satisfied, we observe, that the first term in  $V_t^2$  is equal to its expectation, and thus adds nothing into the overall expectation. Thus, writing

$V_t^2 = \sum_{i=2}^t Y_i$  and using the triangle and Cauchy-Schwarz inequalities,

$$\begin{aligned} \mathcal{E} \left| \frac{V_t^2 - s_t^2}{s_t^2} \right| &= \frac{1}{s_t^2} \mathcal{E} |V_t^2 - s_t^2| = \frac{1}{s_t^2} \mathcal{E} \left| \sum_{i=1}^t (Y_i - \mathcal{E}(Y_i)) \right| \leq \frac{1}{s_t^2} \sum_{i=1}^t \mathcal{E} |Y_i - \mathcal{E}(Y_i)| \\ &\leq \frac{1}{s_t^2} \sum_{i=1}^t \sqrt{\mathcal{E}(Y_i - \mathcal{E}(Y_i))^2} = \frac{1}{s_t^2} \sum_{i=1}^t \sqrt{\text{Var}(Y_i)}. \end{aligned}$$

Making use of the fact that  $\mathcal{E}(m_i) = p\mathcal{E}(M_i)$ ,  $\text{Var}(Y_i)$  involves moments of  $M_i$  and  $U_i$  up to the sixth mixed moments. All of these moments are  $O(t)$  as shown in Appendix A. We

further note  $s_t^2$  involves moments of  $M_i$  and  $U_i$  up to the fourth mixed moments. We are

left with the observation that the desired expectation is  $\frac{O(\sqrt{t})}{O(t)}$  which approaches 0 as

$t \rightarrow \infty$ .

To show condition 2 is satisfied, it suffices to show  $P(|X_i| \geq \epsilon s_t) \rightarrow 0$  as  $t \rightarrow \infty$ ,

since  $\sum \mathcal{E}(X_i^2) = s_t^2$  and  $\mathcal{E}(I(|X_i| > \epsilon s_t)) = P(|X_i| > \epsilon s_t)$ . Using a conditional version

of Markov's inequality (Feller 1966 p. 240) we have  $P(|X_i| \geq \epsilon s_t) \leq \frac{\mathcal{E}(X_i^2)}{\epsilon^2 s_t^2} \rightarrow 0$  as

$t \rightarrow \infty$ , since  $s_t^2$  is  $O(t)$  and  $\mathcal{E}(X_i^2)$  is a single term in the sum. This completes the proof of

Theorem 3.2.

## CHAPTER 4

This chapter will examine performance of the proposed estimating function as demonstrated in a computer simulation, as well as compare performance to other estimators with some actual data sets. Programming was done in FORTRAN on a Digital Equipment Corp. VAX 7620 computer. Subroutines and functions from Numerical Recipes: the Art of Scientific Computing (Press et al 1986) were used in generating binomial random variates for both capture and survival as well as minimizing the conditional least squares function. The actual code is exhibited in Appendix B. It is fully portable to any computer equipped with a standard FORTRAN compiler.

The AMOEBA subroutine used in minimizing the conditional least squares function requires a starting simplex of  $p+1$  points for  $p$  parameters. Starting values are derived from the data making use of the conditional expected values. Since we have

$$\mathcal{E}(u_1) = Np$$

$$\mathcal{E}(u_2 | u_1) = p\phi(N - u_1)$$

$$\mathcal{E}(m_2 | u_1) = p\phi u_1$$

$$\mathcal{E}(m_3 | u_1, u_2) = p\phi(\leq u_1 + u_2)$$

we can solve these equations for the parameters using observed values in place of the expectations and obtain

$$\begin{aligned}\hat{\phi} &= \frac{m_3 - \frac{m_2 u_2}{u_1}}{m_2} \\ \hat{p} &= \frac{m_2}{u_1 \hat{\phi}} \\ \hat{N} &= \frac{u_1}{\hat{p}} \\ \hat{\lambda} &= \frac{u_2 - \hat{p} \hat{\phi} (\hat{N} - u_1)}{\hat{N} \hat{p}}.\end{aligned}$$

These values are then shifted both in the positive and negative directions to arrive at the starting simplex.

#### 4.1 Simulation Results.

Performance of the estimators appears good. As might be expected, in situations with low capture probabilities large standard deviations occur; variation in the estimates decreases as capture probability increases. Figures 4.1 through 4.4 demonstrate the performance of the estimating function for various choices of  $p$ ,  $\phi$ , and  $\lambda$ . In all simulations the initial population was taken to be 1000. In these graphs we see that the average of estimates for both  $N$  and  $p$  are always extremely close to the desired quantity. There appears to be some possible negative bias in the estimation of  $\phi$  and  $\lambda$  but in most cases the 95% confidence intervals do hold the true value. Figures 4.5 and 4.6 demonstrate the lack of an effect due to the number of capture occasions, both in terms of changing the averages of the estimates and changing confidence interval widths. Other combinations of the parameters have been tried in the simulation with similar results.

Figure 4.1 Estimation of the initial population. The true value in all cases is  $N = 1000$ . All are based on 100 repetitions of a ten capture occasion experiment.

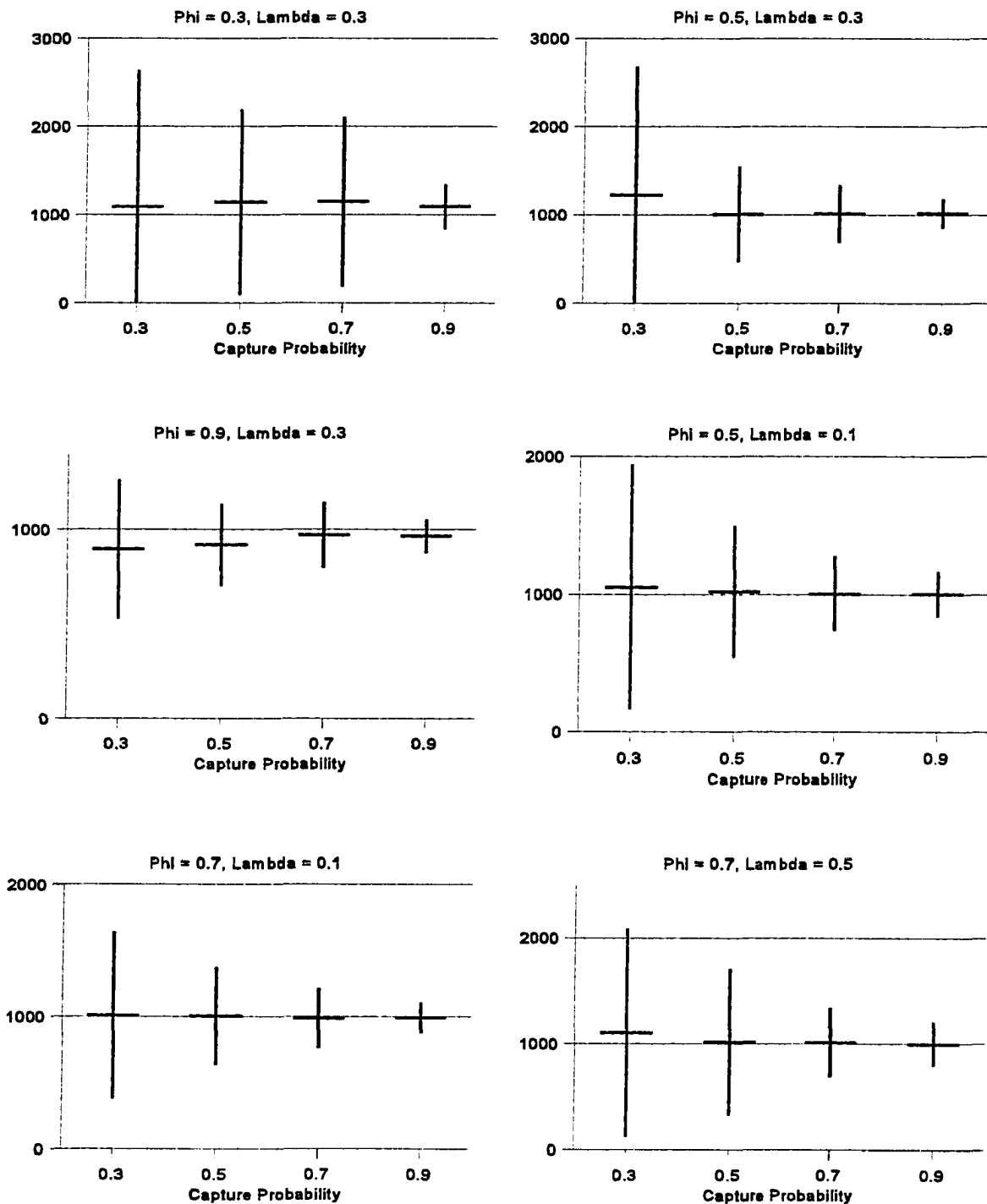




Figure 4.2. Estimation of  $\lambda$ . All figures are based on  $N = 1000$  and 100 repetitions of a 10 capture occasion experiment.

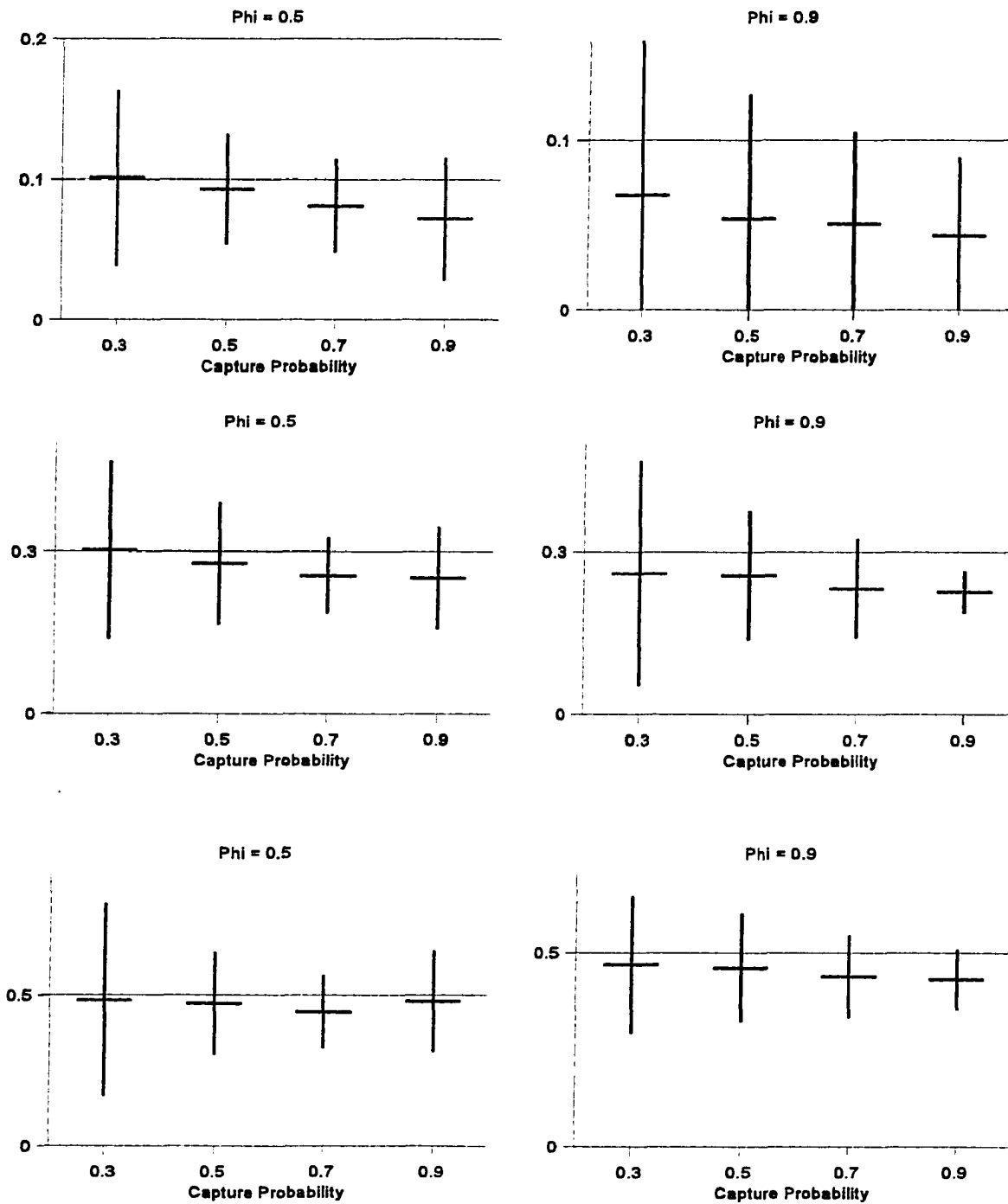


Figure 4.3. Estimation of  $\phi$ . 95% confidence intervals for various levels of  $p$  and  $\lambda$ .

All are based on  $N = 1000$  and 100 repetitions of a ten capture occasion experiment.

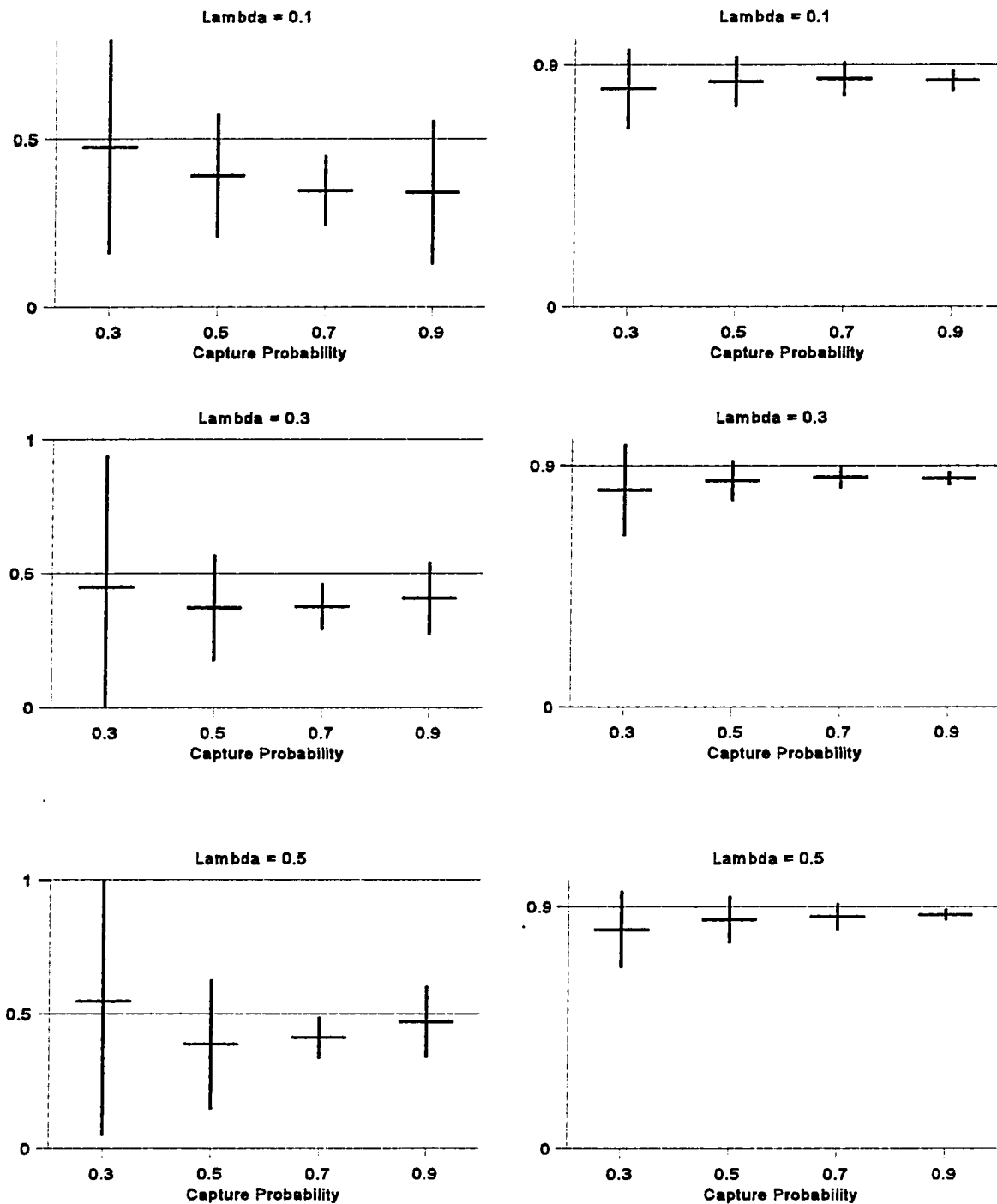


Figure 4.5 Effect of the number of capture occasions on estimation of  $N$  and  $\lambda$ .

All are based on  $N = 1000$ ,  $\lambda = 0.3$ ,  $\phi = 0.7$  and 100 repetitions of the capture experiment.

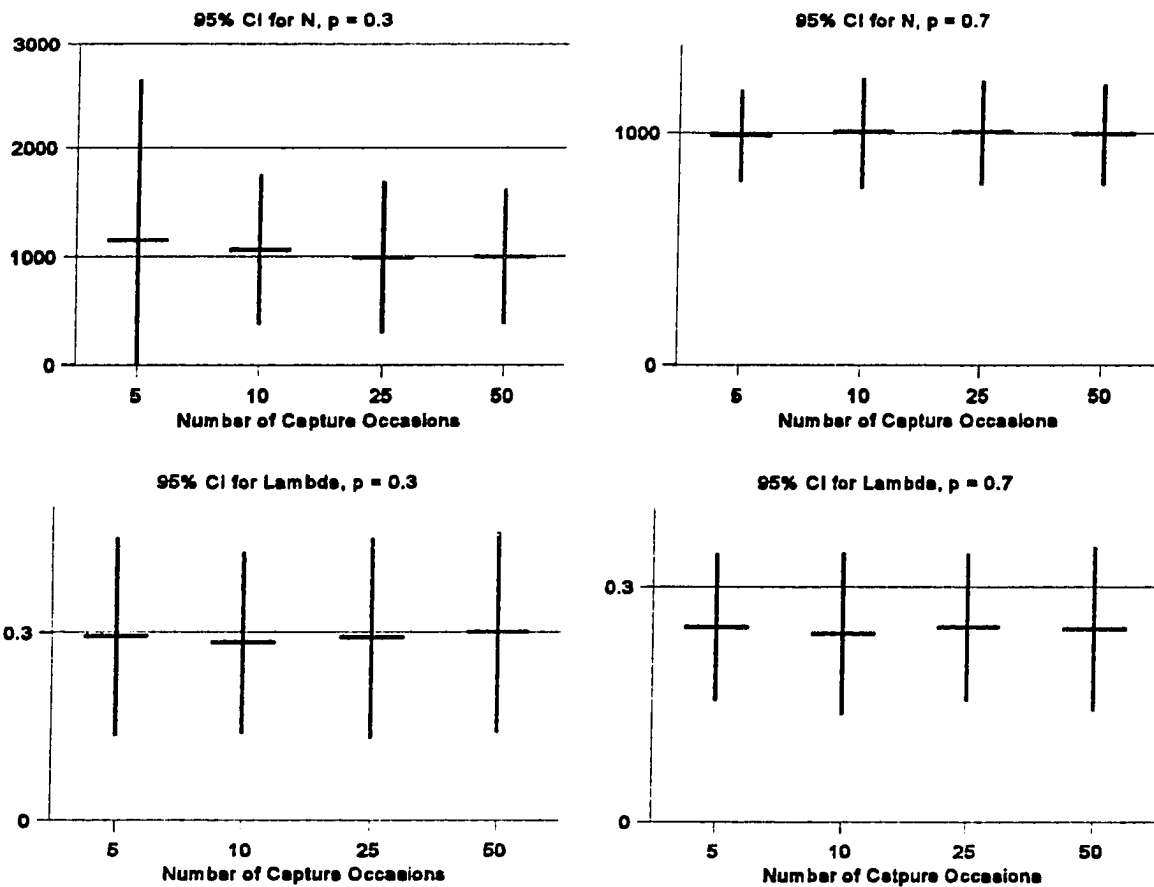


Figure 4.5 Effect of the number of capture occasions on estimation of parameters.

All are based on  $N = 1000$ ,  $\lambda = 0.3$ ,  $\phi = 0.7$

and 100 repetitions of the capture

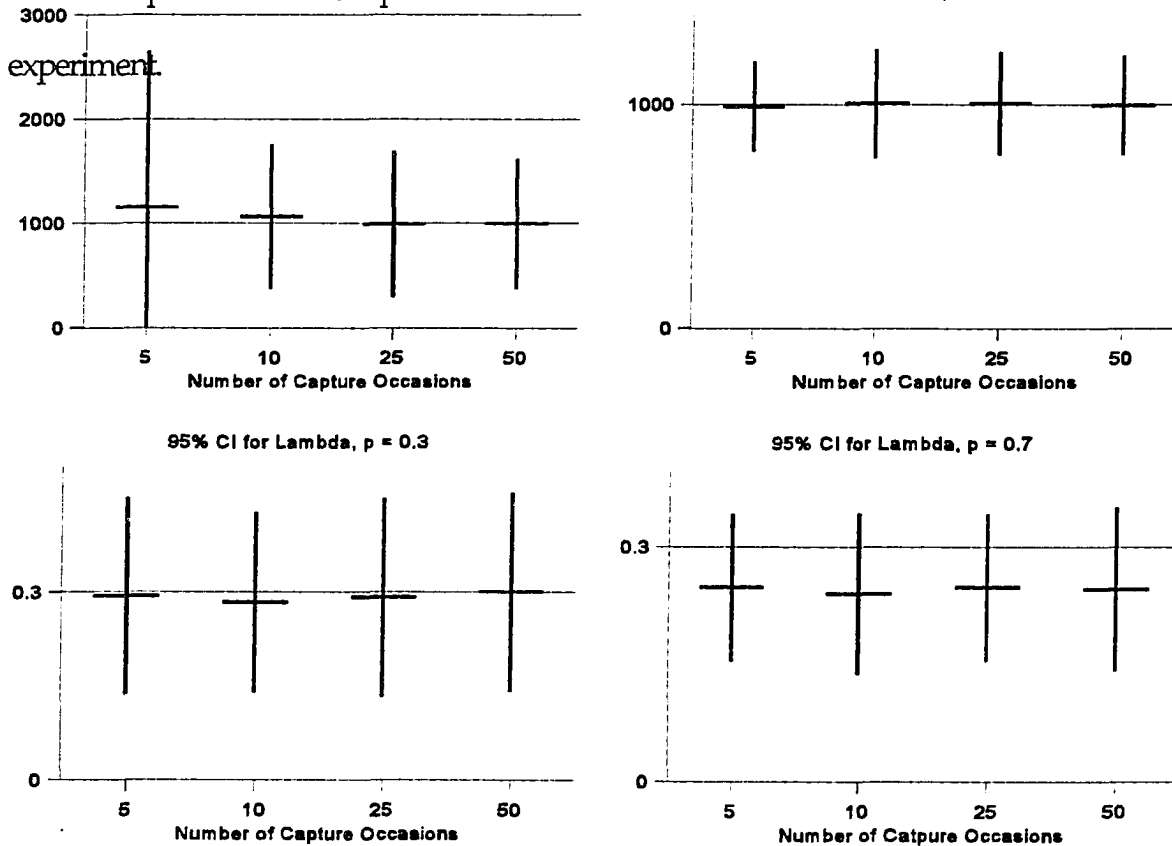
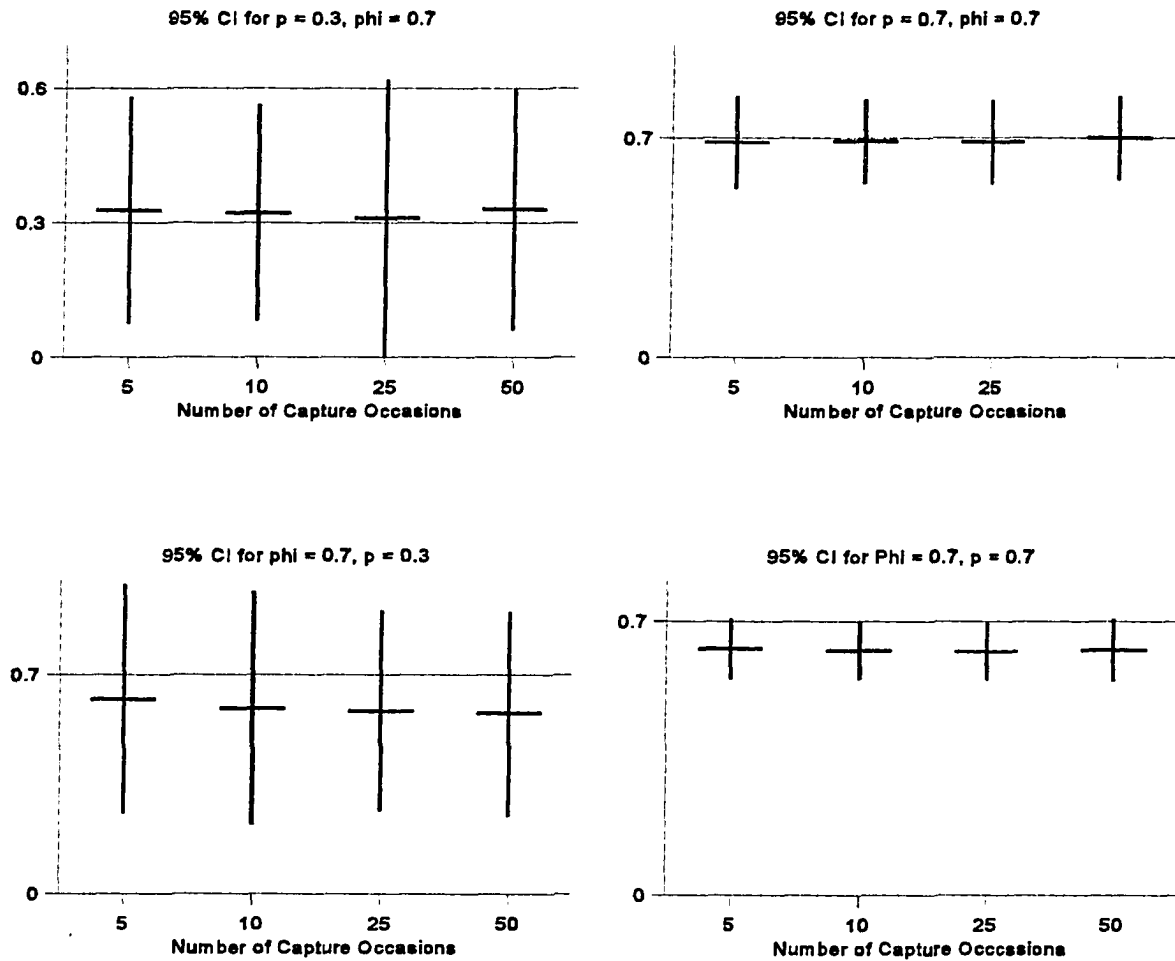


Figure 4.6. Effect of number of capture occasions on estimation of  $p$  and  $\phi$ . All are based on  $N = 1000$ ,  $\lambda = 0.3$  and 100 repetitions of the capture experiment.



#### 4.2 Comparison with other estimators.

It is difficult to fully compare this method of estimation with other open population estimators as the assumption of constant recruitment between capture occasions is extremely restrictive. I have compared it with the Jolly-Seber method, even though this estimator allows for differing recruitment, capture, and survival rates between capture occasions.

The first data set used was originally collected by J. D. Nichols in 1981 and is as cited by Pollock et al (1990 p29). Meadow voles (*Microtus pennsylvanicus*) were trapped at Patuxent Wildlife Research Center, Laurel, Maryland over five day periods at monthly intervals from June to December 1981. This data set was selected as the closest to fulfilling the assumptions of our model.

<b>Table 4.1 Summary Statistics for Meadow Vole data.</b>		
Period	$u_i$	$m_i$
27 Jun - 1 Jul	108	0
1 Aug - 5 Aug	43	84
29 Aug - 2 Sep	29	73
3 Oct - 7 Oct	30	73
31 Oct - 4 Nov	41	61
4 Dec - 8 Dec	60	89

**Table 4.2 Estimates for Meadow Vole Data****Jolly-Seber followed by Proposed Model**

Period	$\phi$	SE	p	SE	N	SE	B	SE
27 Jun-1 Jul	0.88	0.039	-	-	-	-	-	-
1 Aug-5 Aug	0.66	0.048	0.917	0.032	138.4	4.14	30.9	3.50
29 Aug-2 Sep	0.69	0.049	0.863	0.054	118.1	4.41	28.6	2.84
3 Oct-7 Oct	0.63	0.049	0.941	0.025	109.4	2.93	48.3	2.98
31 Oct-4 Nov	-	-	0.917	0.035	111.2	3.13	-	-
4 Dec-8 Dec	-	-	-	-	-	-	-	-
Proposed	0.693	0.130	0.695	0.170	165	54.38	34	14.11

We see from the above results fairly good agreement between the two methods. It should be noted, however, that all capture periods experienced losses on capture varying from one to six animals. This violates one of our basic assumptions and will affect results. In this case it is the most likely cause for capture probability to be estimated lower than in the Jolly-Seber and  $N_i$  being estimated higher.

The second data set is also from Pollock et al (1990, p27). Gray squirrels (*Sciurus carolinensis*) were captured on an approximately monthly basis in a mature oak woodland at Alice Holt Forest Research Station, Surrey, England between November 1972 and September 1974. Five periods in the middle of the experiment (August - September 1973) have been deleted from the data set in fitting both the Jolly-Seber and proposed models

due to extremely low capture numbers. Remaining data used in fitting the models are given below in Table 4.3

Table 4.3 Summary Statistics for Gray Squirrel Data		
Date	$u_i$	$m_i$
Nov. 1972	46	0
Dec. 1972	4	42
Jan. 1973	6	42
Feb. 1973	4	42
Mar. 1973	5	46
Apr. 1973	0	37
May 1973	0	41
May-Jun 1973	3	39
Jun. 1973	4	43
Jul. 1973	5	26
(5 periods deleted)	-	-
Jan. 1974	2	17
Feb. 1974	5	14
Mar. 1974	7	20
Apr. 1974	0	36
May 1974	11	34
Jul. 1974	28	46
Aug. 1974	2	20
Sep. 1974	1	2



**Table 4.4 Estimates for Gray Squirrel data****Jolly-Seber followed by Proposed Model**

Period	$\phi$	SE	p	SE	N	SE	B	SE
Nov. 1972	0.94	0.037	-	-	-	-	-	-
Dec. 1972	0.96	0.030	0.976	0.008	47.1	0.39	6.3	0.77
Jan. 1973	1.00	0.004	0.935	0.021	51.3	0.70	4.5	1.27
Feb. 1973	0.99	0.023	0.821	0.064	56.0	1.19	5.1	1.53
Mar. 1973	0.94	0.041	0.843	0.045	60.5	1.51	-0.8	1.06
Apr. 1973	0.95	0.038	0.674	0.128	54.9	1.23	0.0	0.00
May 1973	1.01	0.030	0.784	0.057	52.3	0.60	4.0	1.22
May-Jun	0.90	0.0552	0.743	0.064	56.5	2.06	3.7	1.45
Jun. 1973	0.92	0.067	0.861	0.041	54.6	1.57	8.7	3.30
Jul. 1973	0.91	0.066	0.526	0.081	58.9	4.59	1.8	6.60
Jan. 1974	0.98	0.068	0.344	0.064	55.2	4.30	13.2	8.25
Feb. 1974	1.02	0.071	0.286	0.060	66.4	8.14	6.9	10.14
Mar. 1974	0.93	0.067	0.362	0.071	74.5	7.91	-10.6	6.28
Apr. 1974	0.99	0.071	0.616	0.079	58.4	2.13	18.2	4.22
May 1974	1.02	0.168	0.592	0.082	76.0	6.12	34.1	8.86
Jul. 1974	0.21	0.048	0.673	0.078	110.3	18.10	-0.6	2.23
Aug. 1974	-	-	1.005	0.002	21.9	0.00	-	-
Sep. 1974	-	-	-	-	-	-	-	-
Proposed	0.777	0.0669	0.764	0.111	63	10.81	20	6.82

This data set is interesting because of the low number of unmarked captures in every period after the first, with the exception of July 1974. Also, the number of recaptures is

very high in relation to total captures indicating a high survival rate. Fitting of both models gives the estimates in table 4.4 above. Notice that for the Jolly-Seber model, there are three periods with estimated negative birth rates, three periods with survival probability estimated as greater than 1.0 and one period with estimated capture probability higher than 1.0. Survival probability estimates are high throughout with the exception of the last one estimated, but capture probability estimates are much lower in the second half of the data than in the first half. The proposed model gives a survival probability lower than that of the Jolly-Seber and a capture probability slightly higher than the average of the Jolly-Seber estimates (0.690). In terms of population numbers, the proposed model indicates a steady expected growth in the population of 3 to 4 animals each period while the Jolly-Seber estimates fluctuate dramatically.

## CHAPTER 5

This paper has presented a proposed model for capture-recapture experiments based on martingales and conditional least squares for a specific type of open population. This is an extension of prior work to a new area. While not optimal in the sense of narrowest confidence intervals (indeed no claim to this effect has been made), it does perform well in estimating population parameters in computer simulation when the assumptions of the model are met.

No model has yet been found to be fully adequate. The model proposed in this paper is certainly not so simply because of its restrictions. Below, some obvious extensions are suggested which would be of use in practical applications.

There are many instances in which it is reasonable to assume relatively constant capture and survival rates but nonconstant recruitment; these pose an immediately obvious extension for future investigation. Such a model should be directly comparable to that of Jolly (1982). Due to the increase in number of parameters, it is certain that some of the field simplicity of this model would be lost. At best, batch marks would be required indicating a marked animal to have first been captured at a particular prior occasion.

Analogous to the above, allowing the recruitment to have some form of distribution (perhaps binomial?) with respect to  $N_i$  would be reasonable and

would require no additional parameters.

Another desirable extension would be to allow for differences in animal behavior due to capture, analogous to the closed population  $M_b$  model. This allows for the “trap happy” or “trap shy” response which is commonly observed. This extension would require one new parameter - the different capture probability for marked animals which is assumed different from that of unmarked animals.

A perhaps simpler extension would allow for losses on capture. If  $R_i$  is the number of animals captured at occasion  $i$  which are actually released, we would change  $\mathcal{L}(M_i)$  (and hence  $\mathcal{L}(m_i)$ ) from being based on prior  $u_i$  to being based on prior  $R_i$ .

Another desirable improvement would be explicit formulae for standard error estimation rather than relying on bootstrap simulation as at present. Since observed quantities are not independent, variance estimates would be conditional at best.

I plan to work on these and other problems in the future.

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## APPENDIX A.

### MOMENTS OF $M_i$ and $U_i$

The joint probability generating function for  $M_i$  and  $U_i$  was seen to be

$$f(x, y) = \mathcal{E}(x^{M_i} y^{U_i}) = [K_1(G_1(x, y))K_2(G_2(x, y)) \dots K_{i-2}(G_{i-2}(x, y))]^{N\lambda} (K_{i-1}(G_{i-1}(x, y)))^N$$

with

$$K_j(G_j(x, y)) = 1 - \phi^j + \phi^j(1 - q^j)x + (\phi q^j)y.$$

Notice first that  $K_j(G_j(1, 1)) = 1$ , so  $f(1, 1) = 1$ . The first partial derivatives evaluated at the point  $(1, 1)$  will give  $\mathcal{E}(M_i)$  and  $\mathcal{E}(U_i)$ . Higher and mixed moments will be obtained by further partial derivatives. For easier computation of these partial derivatives, we will use

$$g(x, y) = \ln(f(x, y)) = N\lambda \sum_{j=1}^{i-2} \ln(K_j(G_j(x, y))) + N\ln(K_{i-1}(G_{i-1}(x, y)))$$

and adjust as necessary for having used the  $g(x, y)$  function. We will denote partial derivative with respect to  $x$  as  $f_1$  and  $g_1$ , those with respect to  $y$  as  $f_2$  and  $g_2$ .

#### A.1 First Moments.



$$g_1(x, y) = \frac{f_1(x, y)}{f(x, y)} = N\lambda \sum_{j=1}^{i-2} \frac{\phi^j(1-q^j)}{K_j(G_j(x, y))} + \frac{N\phi^{i-1}(1-q^{i-1})}{K_{i-1}(G_{i-1}(x, y))}.$$

We have  $g_1(1, 1) = f_1(1, 1) = \mathcal{E}(M_i) = N\lambda \sum_{j=1}^{i-2} \phi^j(1-q^j) + N\phi^{i-1}(1-q^{i-1})$ .

Analogously, we have  $g_2(1, 1) = f_2(1, 1) = \mathcal{E}(U_i) = N\lambda \sum_{j=1}^{i-2} (q\phi)^j + N(q\phi)^{i-1}$ .

## A.2 Second moments.

First notice  $f_{11}(1, 1) = \mathcal{E}(M_i(M_i - 1)) = \mathcal{E}(M_i^2) - \mathcal{E}(M_i)$  but from using the  $g$

function, we have

$$g_{11}(x, y) = \frac{f_{11}(x, y)}{f(x, y)} - \frac{f_{11}(x, y)^2}{(f(x, y))^2} = -N\lambda \sum_{j=1}^{i-2} \frac{\phi^{2j}(1-q^{2j})}{K_j(G_j(x, y))} - N \frac{\phi^{2(i-1)}(1-q^{2(i-1)})}{(K_{i-1}(G_{i-1}(x, y)))^2}$$

and  $g_{11}(1, 1) = f_{11}(1, 1) - (f(1, 1))^2 = \mathcal{E}(M_i^2) - \mathcal{E}(M_i) - (\mathcal{E}(M_i))^2$ . So we have

$$\begin{aligned} \mathcal{E}(M_i^2) &= g_{11}(x, y) + \mathcal{E}(M_i) + (\mathcal{E}(M_i))^2 \\ &= N^2\phi^{2(i-1)}(1-q^{2(i-1)}) - N\phi^{2(i-1)}(1-q^{(i-1)})^2 + N\phi^{i-1}(1-q^{i-1}) + N\lambda \sum_{j=1}^{i-2} \phi^j(1-q^j) \\ &\quad - N\lambda \sum_{j=1}^{i-2} \phi^{2j}(1-q^j)^2 + N^2\lambda^2 \left[ \sum_{j=1}^{i-2} \phi^j(1-q^j) \right]^2 + 2N\lambda\phi^{i-1}(1-q^{i-1}) \sum_{j=1}^{i-2} \phi^j(1-q^j). \end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\mathcal{E}(U_i^2) &= N(N-1)(\phi q)^{2(i-1)} + N(\phi q)^{i-1} + N\lambda\phi q \frac{1 - (\phi q)^{i-2}}{1 - \phi q} + N^2\lambda^2\phi^2 q^2 \left( \frac{1 - (\phi q)^{i-2}}{1 - \phi q} \right)^2 \\ &\quad + 2N^2\lambda(\phi q)^i \frac{1 - (\phi q)^{i-2}}{1 - \phi q} - N\lambda(\phi q)^2 \frac{1 - (\phi q)^{2(i-2)}}{1 - (\phi q)^2}.\end{aligned}$$

The first mixed moment,  $\mathcal{E}(M_i U_i)$ , is equal to  $f_{12}(1, 1)$ . Using the  $g$  function, however, we obtain

$$g_{12}(x, y) = \frac{f_{12}(x, y)}{f(x, y)} - \frac{f_1(x, y)f_2(x, y)}{(f(x, y))^2} = -N\lambda \sum_{j=1}^{i-2} \frac{\phi^{2j} q^j (1 - q^j)}{(K_j(G_j(x, y)))^2} - \frac{N\phi^{2(i-1)} q^{i-1} (1 - q^{i-1})}{(K_{i-1}(g_{i-1}(x, y)))^2}$$

Thus, we have

$$\begin{aligned}\mathcal{E}(M_i U_i) &= g_{12}(1, 1) + \mathcal{E}(M_i) \mathcal{E}(U_i) = N^2\lambda^2 \left[ \sum_{j=1}^{i-2} \phi^j (1 - q^j) \right] \left[ \sum_{j=1}^{i-2} (\phi q)^j \right] \\ &\quad + N^2\lambda\phi^{i-1} (1 - q^{i-1}) \sum_{j=1}^{i-2} (\phi q)^j + N^2\lambda(\phi q)^{i-1} \sum_{j=1}^{i-2} \phi^j (1 - q^j) + N^2(\phi q)^i \phi^{i-1} (1 - q^{i-1}) \\ &\quad - N\lambda \sum_{j=1}^{i-2} \phi^{2j} q^j (1 - q^j) - N\phi^{2(i-1)} q^{i-1} (1 - q^{i-1}).\end{aligned}$$

### A.3 Third Moments

There are four third moments to calculate. In an analogous manner to the preceding, we find  $f_{111}(1, 1) = \mathcal{E}(M_i^3) - 3\mathcal{E}(M_i^2) + 2\mathcal{E}(M_i)$  and

$$\begin{aligned}
g_{111}(x, y) &= \frac{f_{111}(x, y)}{f(x, y)} - \frac{3f_1(x, y)f_{11}(x, y)}{(f(x, y))^2} + \frac{2(f_1(x, y))^3}{(f(x, y))^3} \\
&= 2N\lambda \sum_{j=1}^{i-2} \frac{\phi^{3j}(1-q^j)^3}{(K_j(G_j(x, y)))^3} + \frac{2N\phi^{3(i-1)}(1-q^{i-1})^3}{(K_{i-1}(G_{i-1}(x, y)))^3}.
\end{aligned}$$

Unraveling the above, we find

$$\mathcal{E}(M_i^3) = g_{111}(1, 1) + 3g_{11}(1, 1) + 3g_1(1, 1)g_{11}(1, 1) + g_1(1, 1) + 3(g_1(1, 1))^2 + (g_1(1, 1))^3$$

which leads us to

$$\begin{aligned}
\mathcal{E}(M_i^3) &= 2N\lambda \sum_{j=1}^{i-2} \phi^{3j}(1-q^j)^3 + 2N\phi^{3(i-1)}(1-q^{i-1})^3 - 3N\lambda \sum_{j=1}^{i-2} \phi^{2j}(1-q^j)^2 \\
&\quad - 3N\phi^{2(i-1)}(1-q^{i-1})^2 + N\lambda \sum_{j=1}^{i-2} \phi^j(1-q^j) + N\phi^{i-1}(1-q^{i-1}) \\
&\quad + 3[N\lambda \sum_{j=1}^{i-2} \phi^j(1-q^j) + N\phi^{i-1}(1-q^{i-1})]^2 + [N\lambda \sum_{j=1}^{i-2} \phi^j(1-q^j) + N\phi^{i-1}(1-q^{i-1})]^3 \\
&\quad + 3[N\lambda \sum_{j=1}^{i-2} \phi^j(1-q^j) + N\phi^{i-1}(1-q^{i-1})][N\lambda \sum_{j=1}^{i-2} \phi^{2j}(1-q^j)^2 + N\phi^{2(i-1)}(1-q^{i-1})^2].
\end{aligned}$$

In a like manner, we find the other third moments:

$$\begin{aligned}
\mathcal{E}(U_i^3) &= 2N\lambda(\phi q)^3 \frac{1 - (\phi q)^{3(i-2)}}{1 - (\phi q)^3} + 2N(\phi q)^{3(i-1)} - 3N\lambda(\phi q)^2 \frac{1 - (\phi q)^{2(i-2)}}{1 - (\phi q)^2} - 3N(\phi q)^{2(i-1)} \\
&\quad + N\lambda\phi q \frac{1 - (\phi q)^{i-2}}{1 - (\phi q)} + N(\phi q)^{i-1} + 3[N\lambda\phi q \frac{1 - (\phi q)^{i-2}}{1 - (\phi q)} + N(\phi q)^{i-1}]^2 \\
&\quad + [N\lambda\phi q \frac{1 - (\phi q)^{i-2}}{1 - (\phi q)} + N(\phi q)^{i-1}]^3 \\
&\quad - 3[N\lambda\phi q \frac{1 - (\phi q)^{i-2}}{1 - (\phi q)} + N(\phi q)^{i-1}][N\lambda(\phi q)^2 \frac{1 - (\phi q)^{2(i-2)}}{1 - (\phi q)^2} + N(\phi q)^{2(i-1)}];
\end{aligned}$$

#### A.4 Higher Moments

From the above, it is clear that the m-th moment will involve sums of powers of N,  $\phi$ , and q of order m or smaller. Since N is a constant less than infinity, and  $0 < \phi, q < 1$ ,

$$\begin{aligned} \mathcal{E}(M_i U_i^2) &= 2N\lambda \sum_{j=1}^{i-2} \phi^{3j} q^{2j} (1-q^j) + 2N\phi^{3(i-1)} q^{2(i-1)} (1-q^{i-1}) - N\lambda \sum_{j=1}^{i-2} \phi^{2j} q^j (1-q^j) \\ &\quad - N\phi^{2(i-1)} q^{i-1} (1-q^{i-1}) \\ &\quad - 2[N\lambda \sum_{j=1}^{i-2} \phi^{2j} q^j (1-q^j) + N\phi^{2(i-1)} q^{i-1} (1-q^{i-1})][N\lambda \sum_{j=1}^{i-2} (\phi q)^j + N(\phi q)^{i-1}] \\ &\quad - [N\lambda \sum_{j=1}^{i-2} \phi^j (1-q^j) + N\phi^{i-1} (1-q^{i-1})][N\lambda \sum_{j=1}^{i-2} (\phi q)^{2j} + N(\phi q)^{2(i-1)}] \\ &\quad + [N\lambda \sum_{j=1}^{i-2} \phi^j (1-q^j) + N\phi^{i-1} (1-q^{i-1})][N\lambda \sum_{j=1}^{i-2} (\phi q)^j + N(\phi q)^{i-1}] \\ &\quad + [N\lambda \sum_{j=1}^{i-2} \phi^j (1-q^j) + N\phi^{i-1} (1-q^{i-1})][N\lambda \sum_{j=1}^{i-2} (\phi q)^j + N(\phi q)^{i-1}]^2; \end{aligned}$$

$$\begin{aligned} \mathcal{E}(M_i^2 U_i) &= 2N\lambda \sum_{j=1}^{i-2} \phi^{3j} q^j (1-q^j)^2 + 2N\phi^{3(i-1)} q^{i-1} (1-q^{i-1})^2 - N\lambda \sum_{j=1}^{i-2} \phi^{2j} q^j (1-q^j) \\ &\quad - N\phi^{2(i-1)} q^{i-1} (1-q^{i-1}) \\ &\quad - [N\lambda \phi q \frac{1-(\phi q)^{i-2}}{1-\phi q} + N(\phi q)^{i-1}][N\lambda \sum_{j=1}^{i-2} \phi^{2j} (1-q^j)^2 + N\phi^{2(i-1)} (1-q^{i-1})^2] \\ &\quad + [N\lambda \phi q \frac{1-(\phi q)^{i-2}}{1-\phi q} + N(\phi q)^{i-1}][N\lambda \sum_{j=1}^{i-2} \phi^j (1-q^j) + N\phi^{i-1} (1-q^{i-1})] \\ &\quad + [N\lambda \phi q \frac{1-(\phi q)^{i-2}}{1-\phi q} + N(\phi q)^{i-1}][N\lambda \sum_{j=1}^{i-2} \phi^j (1-q^j) + N\phi^{i-1} (1-q^{i-1})]^2 \\ &\quad + 2[N\lambda \phi q \frac{1-(\phi q)^{i-2}}{1-\phi q} + N(\phi q)^{i-1}][N\lambda \sum_{j=1}^{i-2} \phi^{2j} q^j (1-q^j) - N\phi^{2(i-1)} q^{i-1} (1-q^{i-1})] \end{aligned}$$

we can see all moments will be  $O(t)$  as  $t \rightarrow \infty$ . Higher moments (to order 6) have been

calculated for verification although they are not exhibited here.

## APPENDIX B.

## FORTRAN ESTIMATION AND BOOSTRAP CODE

```

c      program bootstrap
c
c      program to estimate population parameters based on actual data.
c      number of capture occasions and marks / unmarks are input for each
c      simulation estimates standard deviations of parameters.
c
c
c
common / sims / cm(25), u(25), nocc
integer*4 n, cmarks, unmarks
parameter (ndim=4,mp=5,np=4)
real*4 xguess(4), x(4), Nmin, Nmax, Nave, Nsq, newp(4),
1 xscale(4), p(mp,np), y(mp), Nlo, Nhi, lamlow, lamhi
idum = -1
c
c      Accept parameters for the simulation
c
13      print 6
6      format(' Input the number of Capture Occasions - < 11')
```

```

    accept 7, nocc
7    format(i2)
    if (nocc .gt. 25) go to 13
10   print 3
3    format(' Input unmarked and marked format i4, i4')
    do 4 i = 1,nocc
    accept 5, u(i), cm(i)
5    format (f4.0, f4.0)
4    continue
14   print 8
8    format(' Input the number of repetitions for bootstrap - < 201')
    accept 2, nrep
2    format (i3)
    if (nrep .gt. 200) go to 14
    open (unit=10,status='new',file='bootstrap.out')
    write (10,800)
800  format(20x,'Unmarked and marked were :')
    do 9, i = 1, nocc
    write (10,803) u(i), cm(i)
803  format('0',30x,f4.0, 2x, f4.0)
9    continue

```

```

write (10,801) Nocc, Nrep
801  format('0',15x, 'N(occasions) = ',i3, ' Repetitions = ',i3)
write (10,802)
802  format('0')
c
c  perform initial fit to data.
c
phihat = (cm(3) - (cm(2)*u(2) / u(1))) / cm(2)
if (phihat .le. 0.0) phihat = 0.2
if (phihat .ge. 1.0) phihat = 0.8
phat = cm(2) / (u(1)*phihat)
if (phat .le. 0.0) phat = 0.2
if (phat .gt. 1.0) phat = 0.8
anhat = u(1) / phat
alamhat = u(2) / phat - phihat*(anhat - u(1))
if (alamhat .le. 0.0) alamhat = u(nocc) / phat
print *, phihat, phat, alamhat, anhat
p(1,1) = phihat - 0.1
p(1,2) = phat - 0.1
p(1,3) = alamhat - 20.0
if (p(1,3) .lt. 0.0) p(1,3) = 20.0

```



$$p(1,4) = \text{anh} - 50.0$$

$$p(2,1) = 0.1 + p(1,1)$$

$$p(2,2) = p(1,2)$$

$$p(2,3) = p(1,3)$$

$$p(2,4) = p(1,4)$$

$$p(3,1) = p(1,1)$$

$$p(3,2) = p(1,2) + 0.1$$

$$p(3,3) = p(1,3)$$

$$p(3,4) = p(1,4)$$

$$p(4,1) = p(1,1)$$

$$p(4,2) = p(1,2)$$

$$p(4,3) = p(1,3) + 100.0$$

$$p(4,4) = p(1,4)$$

$$p(5,1) = p(1,1)$$

$$p(5,2) = p(1,2)$$

$$p(5,3) = p(1,3)$$

$$p(5,4) = p(1,4) + 100.0$$

$$\text{do } 211 \text{ } l=1,5$$

$$\text{do } 212 \text{ } ik=1,4$$

$$212 \quad x(ik) = p(l, ik)$$

$$y(l) = \text{funk}(x)$$

```

211  continue

      ftol = 0.0000001

      call amoeba(p,y,mp,np,ndim,ftol,funk,iter,ilo,iconv)

213  do 214 ii = 1,4

214  newp(ii) = p(ilo,ii)

      do 215 ii = 1,4

        p(1,ii) = newp(ii)

        p(2,ii) = newp(ii)

        p(3,ii) = newp(ii)

        p(4,ii) = newp(ii)

215  p(5,ii) = newp(ii)

        p(2,1) = p(2,1) + 0.2

        p(3,2) = p(3,2) + 0.2

        p(4,3) = p(4,3) + 10.0

        p(5,4) = p(5,4) + 10.0

      do 216 l=1,5

        do 217 ik=1,4

217  x(ik) = p(l,ik)

        y(l) = funk(x)

216  continue

      call amoeba(p,y,mp,np,ndim,ftol,funk,iter,ilo,iconv)

```

c  
c values obtained will be used in the bootstrap simulation

c

```
in = int(x(4))
```

```
psur = x(1)
```

```
pcap = x(2)
```

```
ilamda = int(x(3))
```

```
alambda = x(3)
```

```
print *, psur, pcap, ilamda, in
```

```
Nmin = 5000.0
```

```
Nmax = 0.0
```

```
Nave = 0.0
```

```
Nsq = 0.0
```

```
Pmin = 1.0
```

```
Pmax = 0.0
```

```
Pave = 0.0
```

```
Psq = 0.0
```

```
Surmin = 1.0
```

```
Surmax = 0.0
```

```
Surave = 0.0
```

```
Sursq = 0.0
```

```

Alammin = 2000.0

Alamax = 0.0

Alamave = 0.0

Alamsq = 0.0

do 300 k=1,nrep
do 310 l = 1,10
cm(l) = 0
310  u(l)= 0
c
c      simulate mark / recapture first capture, then survival to the
c      next period
c
do 100, j=1,nocc
c      Recruitment
if (j .gt. 1) unmarks = unmarks + ilamda
if (j .eq. 1) unmarks = IN
c      Capture Portion
if (j .eq. 1) then
cm(1) = bnldev(pcap, unmarks, idum)
cm(1) = 0

```

```

        u(1) = cmarks

        unmarks = unmarks - cmarks

    else

        cm(j) = bnldev(pcap,cmarks,idum)

        u(j) = bnldev(pcap,unmarks,idum)

        cmarks = cmarks + u(j)

        unmarks = unmarks - u(j)

    endif

C      Survival Portion

        cmarks = bnldev(psur,cmarks,idum)

        unmarks = bnldev(psur,unmarks,idum)

100    continue

c      print 901, (u(i),i=1,10)

c      print 901, (cm(i), i=1,10)

901    format (1x,10(1x,f4.0))

c

c      End of simulation portion. Now solve for HATN, HATPSUR, HAT PCAP

c      using the Martingale estimating function

c      Use the conditional expected values to get starting values for

c      the AMOEBA subroutine

c

```

```

if (cm(2) .eq. 0.0) cm(2) = 1.0
if (cm(3) .eq. 0.0) cm(3) = 1.0
phihat = (cm(3) - (cm(2)*u(2) / u(1))) / cm(2)
if (phihat .le. 0.0) phihat = 0.2
if (phihat .ge. 1.0) phihat = 0.8
phat = cm(2) / (u(1)*phihat)
if (phat .gt. 1.0) phat = 0.8
if (phat .le. 0.0) phat = 0.2
anhat = u(1) / phat
alamhat = u(2) / phat - phihat*(anhat - u(1))
if (alamhat .le. 0.0) alamhat = 31
print *, phihat, phat, alamhat, anhat
p(1,1) = phihat - 0.1
p(1,2) = phat - 0.1
p(1,3) = alamhat - 20.0
if (p(1,3) .lt. 0.0) p(1,3) = 20.0
p(1,4) = anhat - 50.0
p(2,1) = 0.2+p(1,1)
p(2,2) = p(1,2)
p(2,3) = p(1,3)
p(2,4) = p(1,4)

```

```

p(3,1) = p(1,1)
p(3,2) = p(1,2)+ 0.2
p(3,3) = p(1,3)
p(3,4) = p(1,4)
p(4,1) = p(1,1)
p(4,2) = p(1,2)
p(4,3) = p(1,3) + 100.0
p(4,4) = p(1,4)
p(5,1) = p(1,1)
p(5,2) = p(1,2)
p(5,3) = p(1,3)
p(5,4) = p(1,4)+100.0
do 311 l=1,5
do 312 ik=1,4
312  x(ik) = p(l,ik)
      y(l) = funk(x)
311  continue
      ftol = 0.0000001
      call amoeba(p,y,mp,np,ndim,ftol,funk,iter,ilo,iconv)
313  do 314 ii = 1,4
314  newp(ii) = p(ilo,ii)

```

```

do 315 ii = 1,4
  p(1,ii) = newp(ii)
  p(2,ii) = newp(ii)
  p(3,ii) = newp(ii)
  p(4,ii) = newp(ii)
315  p(5,ii) = newp(ii)
  p(2,1) = p(2,1) + 0.2
  p(3,2) = p(3,2) + 0.2
  p(4,3) = p(4,3) + 10.0
  p(5,4) = p(5,4) + 10.0
do 316 l=1,5
  do 317 ik=1,4
317  x(ik) = p(l,ik)
    y(l) = funk(x)
316  continue
  call amoeba(p,y,mp,np,ndim,ftol,funk,iter,ilo,iconv)
  if (iconv .eq. 1) go to 319
  itry = 0
  if (p(ilo,1) .gt. 1.0) then
    p(ilo,1) = 0.8
    itry = 1

```



endif

if (p(ilo,1) .lt. 0.0) then

    p(ilo,1) = 0.1

    itry = 1

endif

if (p(ilo,2) .gt. 1.0) then

    p(ilo,2) = 0.8

    itry = 1

endif

if (p(ilo,2) .lt. 0.0) then

    p(ilo,2) = 0.1

    itry = 1

endif

if (p(ilo,3) .gt. 1000.0) then

    p(ilo,3) = 900.0

    itry = 1

endif

if (p(ilo,3) .lt. 0.0) then

    p(ilo,3) = 1.0

    itry = 1

endif

```

if (p(ilo,4) .lt. 0.0) then
    p(ilo,4) = 1.0
    itry = 1
endif

if (itry .eq. 1) go to 313
319 do 318 ii = 1,4
318 x(ii) = p(ilo,ii)
    fvalue = funk(x)
    write (10,900) (x(I),i=1,4), fvalue
900 format (' Phi = ',f7.5,' P = ',f7.5,' Lambda = ',f10.5,
1 ' N = ',f10.5, ' Funk = ',f10.5)
if (X(4) .le. Nmin) Nmin = X(4)
if (X(4) .ge. Nmax) Nmax = X(4)
Nave = Nave + X(4) / float(nrep)
Nsq = Nsq + X(4)**2
if (x(1) .le. Surmin) Surmin = x(1)
if (x(1) .ge. Surmax) Surmax = x(1)
Surave = Surave + x(1) / float(nrep)
Sursq = Sursq + x(1)**2
if (x(2) .le. Pmin) Pmin = x(2)
if (x(2) .ge. Pmax) Pmax = x(2)

```

```

Pave = Pave + x(2) / float(nrep)

Psq = Psq + x(2)**2

if(x(3) .le. Alammin) Alammin = x(3)

if(x(3) .ge. Alammax) Alammax = x(3)

Alamave = Alamave + x(3) / float(nrep)

Alamsq = Alamsq + x(3)**2

300  continue

c

c  End of Repetition Loop. Tie up loose ends and get info
c  on distribution of estimates
c

arep = float(nrep)

SDN = sqrt((Nsq - arep*Nave**2) / (arep - 1.0))

SDSur = sqrt((Sursq - arep * Surave**2) / (arep - 1.0))

SdP = sqrt((Psq - arep * Pave**2) / (arep - 1.0))

SdAlam = sqrt((Alamsq - arep*Alamave**2) / (arep - 1.0))

Nlo = IN - 1.96*SDN

if (Nlo .le. 0.0) Nlo = 0.0

Nhi = IN + 1.96*SDN

Philow = psur - 1.96*SDSur

if (philow .le. 0.0) philow = 0.0

```

```

phihi = psur + 1.96*SDSur
plow  = pcap - 1.96*SdP
if (plow .le. 0.0) plow = 0.0
phi   = pcap + 1.96*SdP
lamlow = float(ilaamda) - 1.96*SdAlam
if (lamlow .le. 0.0) lamlow = 0.0
lamhi  = float(ilaamda) + 1.96*SdAlam
print 31, in, SdN, Nlo, Nhi
31  format(' N hat= ',i8, ' sd(N) = ',f8.4, ' Low 95% = ',f7.3,
1 ' Hi 95% = ', f8.3)
print 32, psur, SDSur, Philow, Phihi
32  format(' Phi hat= ',f7.3, ' sd(Phi) = ',f8.4, ' Low 95% = ',
1 f7.3, ' Hi 95% = ', f7.3)
print 33, Pcap, SdP, Plow, phi
33  format(' P hat= ',f7.3, ' sd(P) = ',f8.4, ' Low 95% = ',f7.3,
1 ' Hi 95% = ', f7.3)
print 34, ilaamda, SdAlam, lamlow, lamhi
34  format(' LambdaN hat = ', i8, ' sd(Lam)= ', f8.4, ' Low 95% = ',
1 f8.3, ' High 95% = ', f8.3)
write (10,31) in, SdN, Nlo, Nhi
write (10,32) Surave, SDSur, Philow, Phihi

```

```

write (10,33) Pave, SdP, Plow, phi
write (10,34) ilamda, SdAlam, lamlow, lamhi

```

```
860  continue
```

```
850  continue
```

```
close(unit=10,disp='keep')
```

```
end
```

```
c
```

```
c  Function to evaluate the objective function - required by AMOEBA
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c
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function funk(x)
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common / sims / cm(25), u(25), nocc
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integer n
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real x(4)
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c
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c  x(1) = Phi  (Survival Probability)
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c  x(2) = P    (Capture Probability)
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c  x(3) = Lambda (Recruitment parameter)
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c  x(4) = N    (Initial population)
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c
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f = 0.0
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    f = f + (u(1) - x(2)*x(4))**2

    do 10 j= 2, nocc

    k = j-1

    a = cm(j) - u(j) - x(1)*cm(k) + x(1)*(1.0 - 2.0*x(2))*u(k)

    a = a + x(2)*x(3)

10    funk = f + a**2

    return

end

FUNCTION ran1(idum)

INTEGER idum,IA,IM,IQ,IR,NTAB,NDIV

REAL ran1,AM,EPS,RNMX

PARAMETER (IA=16807,IM=2147483647,AM=1. / IM,IQ=127773,IR=2836,
*NTAB=32,NDIV=1+(IM-1) / NTAB,EPS=1.2e-7,RNMX=1.-EPS)

INTEGER j,k,iv(NTAB),iy

SAVE iv,iy

DATA iv /NTAB*0/, iy /0/

if (idum.le.0.or.iy.eq.0) then

    idum=max(-idum,1)

    do 11 j=NTAB+8,1,-1

    k=idum / IQ

    idum=IA*(idum-k*IQ)-IR*k

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      if (idum.lt.0) idum=idum+IM
      if (j.le.NTAB) iv(j)=idum
11  continue
      iy=iv(1)
      endif
      k=idum/IQ
      idum=IA*(idum-k*IQ)-IR*k
      if (idum.lt.0) idum=idum+IM
      j=1+iy/NDIV
      iy=iv(j)
      iv(j)=idum
      ran1=min(AM*iy,RNMX)
      return
      END

```

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SUBROUTINE amoeba(p,y,mp,np,ndim,ftol,funk,iter,ilo,iconv)
  INTEGER iter,mp,ndim,np,NMAX,ITMAX
  PARAMETER (NMAX=20,ITMAX=10000,alpha=1.0,beta=0.5,gamma=2.0)
  REAL ftol,p(mp,np),y(mp),funk,pr(nmax),prr(nmax),PBAR(NMAX)
  EXTERNAL funk

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CU USES amotry,funk

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INTEGER i,ih,i,ilo,inhi,j,m,n

REAL rtol,sum,swap,ysave,ytry,psum(NMAX),amotry

      mpts = ndim+1

      iter=0

      iconv = 0

1      ilo = 1

      if(y(1).gt. y(2))then

          ihi = 1

          inhi = 2

      else

          ihi = 2

          inhi = 1

      endif

do 11 i=1,mpts

    if(y(i).lt.y(ilo)) ilo=i

    if(y(i).gt.y(ihi)) then

        inhi=ihi

        ihi=i

    else if(y(i).gt.y(inhi)) then

        if(i.ne.ihi) inhi=i

    endif

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11  continue

    rtol=2.*abs(y(ihi)-y(ilo)) / (abs(y(ihi))+abs(y(ilo)))

    if (rtol.lt.ftol) return

    if (y(ilo) .le. ftol) return

    if (iter .eq. itmax) then

        print 2,rtol, y(ilo), y(ihi)

2      format (' rtol = ',f10.5, ' y(hi) = ', f11.8, ' y(lo) = ',f11.8)

        print 3,(p(ilo,j),j=1,4)

3      format (' params = ', 4(1x,f10.4))

        iconv = 1

        return

    endif

    mpts = ndim + 1

    iter = iter + 1

    do 12, j = 1,ndim

        pbar(j) = 0.

12     continue

        do 14 i=1,mpts

            if (i .ne. ihi) then

                do 13 j=1,ndim

                    pbar(j) = pbar(j) + p(i,j)

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13         continue

        endif

14     continue

    do 15 j=1,ndim

        pbar(j)=pbar(j)/ndim

        pr(j) = (1. + alpha)*pbar(j) - alpha*p(ihi,j)

15     continue

    ypr = funk(pr)

    if (ypr .le. y(ilo)) then

        do 16 j=1,ndim

16         prr(j) = gamma*pr(j)+(1. - gamma)*pbar(j)

        yprr = funk(prr)

        if (yprr .lt. y(ilo)) then

            do 17 j=1,ndim

17                 p(ihi,j) = prr(j)

                y(ihi) = yprr

            else

                do 18 j=1,ndim

18                 p(ihi,j) = pr(j)

                y(ihi) = ypr

            endif

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else if(ypr.ge. y(inhi)) then
    if(ypr.lt. y(ihi)) then
        do 19 j=1,ndim
19      p(ihi,j) = pr(j)
        y(ihi) = ypr
    endif
    do 21 j=1,ndim
21    prr(j) = beta*p(ihi,j)+ (1.-beta)*pbar(j)
    ypr = funk(prr)
    if (yprr.lt. y(ihi)) then
        do 22 j = 1,ndim
22      p(ihi,j) = prr(j)
        y(ihi) = yprr
    else
        do 24 i=1,mpts
        if (i.ne. ilo) then
            do 23 j=1,ndim
                pr(j) = 0.5*(p(i,j)+p(ilo,j))
23      p(i,j) = pr(j)
                y(i) = funk(pr)
            endif
        enddo
    endif

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24          continue

          endif

      else

          do 25 j=1,ndim
25      p(ihi,j) = pr(j)
          y(ihi) = ypr

          endif

      go to 1

  END

```

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```

FUNCTION bnldev(pp,n,idum)

  INTEGER idum,n

  REAL bnldev,pp,PI

CU  USES gammln,ran1

  PARAMETER (PI=3.141592654)

  INTEGER j,nold

  REAL am,em,en,g,oldg,p,pc,pclog,plog,pold,sq,t,y,gammln,ran1

  SAVE nold,pold,pc,plog,pclog,en,oldg

  DATA nold /-1/, pold /-1./

  if(pp.le.0.5)then

    p=pp

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else
  p=1.-pp
endif
am=n*p
if (n.lt.25)then
  bnldev=0.
  do 11 j=1,n
    if(ran1(idum).lt.p)bnldev=bnldev+1.
11  continue
  else if (am.lt.1.) then
    g=exp(-am)
    t=1.
    do 12 j=0,n
      t=t*ran1(idum)
      if (t.lt.g) goto 1
12  continue
    j=n
1  bnldev=j
  else
    if (n.ne.nold) then
      en=n

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    oldg=gammln(en+1.)
    nold=n
endif
if (p.ne.pold) then
    pc=1.-p
    plog=log(p)
    pclog=log(pc)
    pold=p
endif

sq=sqrt(2.*am*pc)
2  y=tan(PI*ran1(idum))

em=sq*y+am

if (em.lt.0..or.em.ge.en+1.) goto 2

em=int(em)

t=1.2*sq*(1.+y**2)*exp(oldg-gammln(em+1.)-gammln(en-em+1.))+em*
*plog+(en-em)*pclog

if (ran1(idum).gt.t) goto 2

bnldev=em

endif

if (p.ne.pp) bnldev=n-bnldev

return

```

END

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FUNCTION gammln(xx)

REAL gammln,xx

INTEGER j

DOUBLE PRECISION ser,stp,tmp,x,y,cof(6)

SAVE cof,stp

DATA cof,stp / 76.18009172947146d0,-86.50532032941677d0,

\*24.01409824083091d0,-1.231739572450155d0,.1208650973866179d-2,

\*-.5395239384953d-5,2.5066282746310005d0 /

x=xx

y=x

tmp=x+5.5d0

tmp=(x+0.5d0)\*log(tmp)-tmp

ser=1.000000000190015d0

do 11 j=1,6

y=y+1.d0

ser=ser+cof(j) / y

11 continue

gammln=tmp+log(stp\*ser / x)

return

END

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